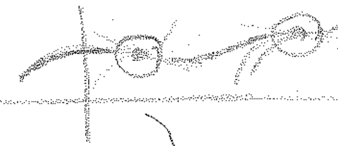
DAY 3
Mon Aug 31

CONVERGENCE, EXPANSION POINTS



"PROBLEMATIC" Taylor Series:

- $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} x^k / k$ — no factorial! — $k=1$: not zero!
- $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$
- $\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{2k+1}$
- "Binomial" $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$

Unlike e^x, $\cos x$, $\sin x$, $\cosh x$, $\sinh x$, these ONLY hold for $|x|<1$, (or $-1 < x < +1$)

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

Q1 What is the Taylor series (about 0) of $f(x) = \dfrac{x}{(1+2x)^{3/2}}$, and where does it converge?

Ans Rewrite as $x(1+2x)^{-3/2}$, use the binomial series for $(1+2x)^{-3/2}$:

$$f(x) = x\left[\sum_{k=0}^{\infty} \binom{-3/2}{k} (2x)^k\right]$$

$$= \sum_{k=0}^{\infty} \binom{-3/2}{k} 2^k \cdot x^{k+1}$$

This works for $|2x| < 1$, or $|x| < \frac{1}{2}$

BUT what are the terms?

k

0. $\binom{-3/2}{0} = 1$

1. $\binom{-3/2}{1} = \frac{-3/2}{1} = -3/2$

2. $\binom{-3/2}{2} = \frac{(-3/2)(-3/2-1)}{2!} = \frac{(-3/2)(-5/2)}{2} = \frac{15}{8}$

3. $\binom{-3/2}{3} = \frac{(-3/2)(-5/2)(-7/2)}{3!} = \frac{-105}{48} = -\frac{35}{16}$

So, $f(x) = x - 3x^2 + \frac{15}{2}x^3 - \frac{35}{2}x^4 + HOT$

(ONLY works for $|x| < 1/2$)

EXPANSION AWAY FROM ZERO

About $x = a$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + HOT$$

Remember This? IMPORTANT!

You can always plug a=0 to get the old setting back!

Q2 Use a linear approximation to estimate the value of $\sqrt{103}$

Ans. We want to expand $f(x) = \sqrt{x}$ about $x = 100$, because we know $\sqrt{100} = 10$ AND 100 IS near 103. So:

$$f(x) = \sqrt{x}, \quad f(100) = \sqrt{100} = 10.$$

$$f'(x) = +1/2\sqrt{x}, \quad f'(100) = +1/20.$$

By Taylor-magic, near $a = 100$ we have

$$f(x) \approx 100 + \frac{(x-100)}{20} + \text{H.O.T.},$$

so $f(103) \approx 100 + 3/20 = \underline{\underline{100.15}}$

(FYI, $\sqrt{103} \approx 10.14889$).

⚡ BEWARE COMPOSITION !!

Q3 Expand about $x = 0$ up to degree $\underline{4}$.

A) $\cos(\pi \sin x)$ B) $\sin(\pi \cos x)$

Ans $\underline{\underline{A}}$: $\cos(\pi \sin x) = 1 - \frac{(\pi \sin x)^2}{2!} + \frac{(\pi \sin x)^4}{4!} - \text{HOT}$

$$= 1 - \frac{\pi^2}{2!}\left(x - x^3/3! + \text{HOT}\right)^2 + \frac{\pi^4}{4!}\left(x - x^3/3! + \text{HOT}\right)^4 - \text{HOT}$$

$$= 1 - \frac{\pi^2}{2!}x^2 + \left(2\pi^2/3! + \pi^4/4!\right)x^4 + \text{HOT} \quad \checkmark$$

Let's Try.!

B $\sin(\pi \cos x) = \pi \cos x - \frac{(\pi \cos x)^3}{3!} + \frac{(\pi \cos x)^5}{5!} - HOT$

$$= \pi\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + HOT\right) - \frac{\pi^3}{3!}\left(1 - \frac{x^2}{2!} + HOT\right)^3$$

$$+ \pi^5/5!\left(1 - x^2/2! + HOT\right)^5 - HOT.$$

Aaaah!! The zero-order term is an infinite sum: $\left(\pi - \pi^3/3! + \pi^5/5! - \cdots\right)$.

WHAT WENT WRONG??

- Well, at $x=0$: $\pi \cos x = \pi$, So we need to expand sin at π, not 0
- Take derivatives:

0 $\sin(\pi) = 0$
1 $\cos(\pi) = -1$
2 $-\sin(\pi) = 0$
3 $-\cos(\pi) = 1$

near π,
$$\sin(x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(x-\pi)^{2k+1}}{(2k+1)!}$$