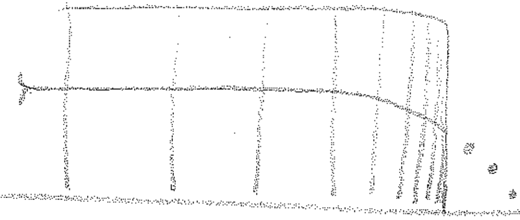
DAY 4
Wed Sep 2

LIMITS



IN

- A function $f(x)$ →
- A number a →

lim

OUT

→ A number $\lim\limits_{x\to a} f(x)$ — The "limit of f at a"

(Does NOT always exist!)

Def

Very, very IMPORTANT

$\lim\limits_{x\to a} f(x) = L$ if

- for every $\varepsilon > 0$ ← chosen by the "enemy"
- there is a $\delta > 0$ ← chosen by you!

so that $|x-a| < \delta$ forces $|f(x) - f(a)| < \varepsilon$

ALL OF CALCULUS DEPENDS ON THIS DEFINITION...

Notes

These are useful to keep in mind...

We will see examples of all of these

1. This definition does NOT tell you what "L" should be, given $f(x)$ and a.

2. But for most "nice" functions [e^x, $\cos x$, $\sin x$, $\ln x$, x^k, ...] we have
$$\lim_{x\to a} f(x) = f(a) \quad [\text{for } a \text{ in the domain of } f]$$

3. Sometimes we must deal with a or L being + or − infinity!

4. Sometimes, NO SUCH L exists. This is NOT the same as $L = \pm\infty$!

Q1 A) What is $\lim_{x \to 0} (1+x)$?

Ans: Just plug in $x=0$, get $\lim_{x\to 0} 1+x = \boxed{1}$

Q1 B) What is $\lim_{x \to 0} (1+x)^{1/x}$?

Ans. This is HARDER, we can't just plug in $x=0$, since the $1/x$ power is not defined for that value of x. BUT...

This is the only "clever" step, the rest is algebra!

Set $y = (1+x)^{1/x}$ [we want $\lim_{x\to 0} y$]

So, $\ln y = \frac{1}{x} \ln(1+x)$

And now, TAYLORIZE (near $x=0$).

$$\ln y = \tfrac{1}{x}\left[x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \text{HOT}\right]$$

$$= \left[1 - \tfrac{x}{2} + \tfrac{x^2}{3} - \text{HOT}\right]$$

Clearly, $\lim_{x\to 0} \ln y = 1$

Since ln is a "nice" function, we have

$$\ln\left[\lim_{x\to 0} y\right] = 1,$$

(This is not at all obvious!)

so $\boxed{\lim_{x\to 0} y = e}$

Q2 A) What is $\lim_{x\to\pi} \cosh(\sin^2 x)$?

a) $\lim_{x\to\pi} \sin^2 x = \sin^2 \pi = 0$

b) $\lim_{y\to 0} \cosh y = \cosh 0 = \boxed{1}$

$\cosh y = \dfrac{e^y + e^{-y}}{2}$

Q2 B) What is $\lim_{x \to 0} \frac{x \cos x}{\sin 2x}$?

Again, TAYLOR:

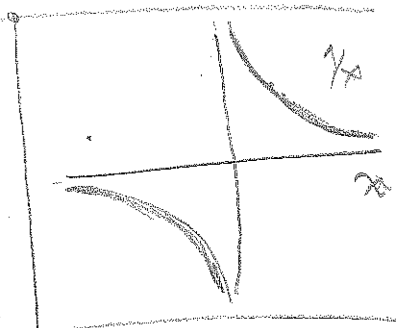
$$x \cos x = x\left(1 - x^2/2! + HOT\right)$$

$$\sin 2x = 2x - 8x^3/3! + HOT.$$

So: $$\frac{x \cos x}{\sin 2x} = \frac{x\left(1 - x^2/2! + HOT\right)}{2x - 8x^3/3! + HOT}$$

$$= \frac{\cancel{x}\left(1 - x^2/2! + HOT\right)}{\cancel{x}\left(2 - 8x^2/3! + HOT\right)}$$

So, $$\boxed{\lim_{x \to 0} \frac{x \cos x}{\sin 2x} = 1/2}$$



Q3 A) What is $\lim_{x \to 0^+} 1/x$?

The answer is $\boxed{+\infty}$. To see this, we must show that for every number N, we can find a small enough $\delta > 0$ so that $1/\delta > N$. So, any $\delta < 1/N$ works.

- Similarly, $\lim_{x \to 0^-} 1/x = \boxed{-\infty}$

- From the two computations above,

$$\lim_{x \to 0} 1/x \text{ does } \underline{\text{NOT}} \text{ exist!}$$

$(+\infty \neq -\infty)$.

Q3 B) What is $\lim_{x \to \infty} 1/x^2$?

Ans. Here we get $\boxed{0}$; note that given any $\varepsilon > 0$ we have $1/x^2 < \varepsilon$ for every $|x| > 1/\sqrt{\varepsilon}$. So for large enough x, we get small enough $1/x^2$

Q4 Define the function $f(x)$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$$

For which "a" does $\lim_{x \to a} f(x)$ exist ?

Ans. Amazingly, $\lim_{x \to a} f(x)$ does NOT exist for ANY real number a! If a is rational, then one can find irrational numbers x with $|x-a| < \delta$ for any δ, but $|f(x) - f(a)| = |-1| = \underline{1}$

And if a is irrational then we can find a rational number x with $|x-a| < \delta$ for any δ, but $|f(x) - f(a)| = |1| = \underline{1}$

So, even for $\varepsilon = 1/2$, there is no δ regardless of which "a" we choose!