DAY 5
Fri Sep 4

L'HÔPITAL'S RULE & ORDERS OF GROWTH

L'HÔP

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

WHY
$$\frac{f(x)}{g(x)} = \frac{f(a) + f'(a)(x-a) + \cdots}{g(a) + g'(a)(x-a) + \cdots}$$
both zero

Sometimes it takes work to get functions in this "0/0" form; but this rule is EXTREMELY useful!

Q1 Compute $\displaystyle\lim_{x\to 0} \frac{\cos(ax) - 1}{\sin(bx)}$ for $a, b \neq 0$.

Ans This already has the form "0/0", so just differentiate away:

$$\lim_{x\to 0} \frac{\cos(ax) - 1}{\sin(bx)} = \lim_{x\to 0} \frac{-a\overbrace{\sin ax}^{\to 0}}{b\underbrace{\cos bx}_{\to b}} = 0$$

Q2 And now, $\displaystyle\lim_{x\to 0} \left[\cot x - \csc x\right]$??

Ans Wait, what? This doesn't look like "0/0"! But wait,

$$\cot x - \csc x = \frac{\cos x}{\sin x} - \frac{1}{\sin x} = \frac{\cos x - 1}{\sin x}$$

Now, use your anwer to Q1 (with $a = b = 1$ ...)

Q3 Fine, genius. Now do $\displaystyle\lim_{x\to 1} \left[1 + \ln x\right]^{2/(x-1)}$

This is also NOT in the "0/0" form,
so let's do what we did on DAY 4:

$$\text{Set } y = (1+\ln x)^{2/(x-1)}$$

$$\text{So, } \ln y = \frac{2\ln(1+\ln x)}{x-1}$$

As $x \to 1$, this DOES have the "0/0" form!

Now,

$$\lim_{x\to 1} \ln y = 2 \lim_{x \to 1} \frac{\ln(1+\ln x)}{x-1}$$

L'Hôp away:

$$\lim_{x\to 1} \ln y = 2 \lim_{x\to 1} \frac{\frac{1}{1+\ln x}\cdot\frac{1}{x}}{1}$$

$$= 2$$

So, $\boxed{\lim_{x\to 1} y = e^2}$

THE BIG O

Def A) (at 0)

We say $f(x)$ is $O(g(x))$ as $x \to 0$ if

$|f(x)| < C|g(x)|$ for some const. C.

for all x "near" 0. $\boxed{\text{i.e. } \lim_{x\to 0}\left|\frac{f(x)}{g(x)}\right| \neq \infty}$

B) (at ∞)

We say $f(x)$ is $O(g(x))$ as $x \to \infty$ if

$|f(x)| < C|g(x)|$ for some const. C

for all "large" x $\boxed{\text{i.e. } \lim_{x\to \infty}\left|\frac{f(x)}{g(x)}\right| \neq \infty}$

O Replaces "H.O.T." ...

As $x \to 0$, we have:

- $\sin(x) = O(x)$
- $\sin(x) = x + O(x^3)$
- $\sin(x) = x - x^3/3! + O(x^5)$, etc

(BUT ALSO...)

- $\sin(x) = O(1)$ $\quad \lim \frac{\sin x}{1} \neq \infty$
- $\sin(x) = x + O(x^2)$
- $\sin(x) = x - x^3/3! + O(x^4)$ $\leftarrow$ Because $\lim_{x \to 0} \frac{x^5}{x^4} \neq \infty$

VERY Useful:

a) $O(f(x)) \cdot O(g(x)) = O(f(x) \cdot g(x))$

b) $O(f(x)) + O(g(x)) = O(f(x)) + O(g(x))$

c) If $f(x) = O(g(x))$ then $O(f(x)) = O(g(x))$

d) $O(k f(x)) = O(f(x))$ for any constant k.

Q4

Show that x^n is $O(x^x)$ for every $n > 1$.
as $x \to \infty$

Ans.

For $x > n$, we have:

$$|x^n| < 1 \cdot |x^x| \quad \text{so } \underline{\text{DONE}}$$

Large enough x means $x^n < (\text{const})\, x^x$

Q5

Find $f(x)$ so that $\sin^2 x \ln(1+x) = f(x) + O(x^3)$
as $x \to 0$

Ans

$$\sin^2 x \ln(1+x) = \left(x - x^3/3! + O(x^5)\right)\left(x - x^2/2 + O(x^3)\right)$$

(near 0)

$$= x^2 - x^3/3 + x\,O(x^3) - x^4/3! + x^5/2 \cdot 3!$$
$$- x^3/3!\, O(x^3) + x\,O(x^5) - x^2/2\, O(x^5)$$
$$+ O(x^5)\, O(x^3)$$

This immediately simplifies to $x^2 + O(x^3)$,

So: $\boxed{f(x) = x^2}$

How??

- $x^3/3!$ is clearly $O(x^3)$ as $x \to 0$
- $x\,O(x^3) = O(x^4) = O(x^3)$ as $x \to 0$ because $x^4 = O(x^3)$
- cx^n is $O(x^3)$ for any $n \geq 3$!
- $O(x^3)\,O(x^5) = O(x^8)$, etc.

Q5. If $f(x) = x^6 + 3x^{10}$, find:

a) The largest n so $f(x) = O(x^n)$ as $x \to 0$

b) The smallest m so $f(x) = O(x^m)$ as $x \to \infty$.

Ans a).

$$\lim_{x\to 0} \frac{f(x)}{x^n} = \lim_{x\to 0}\left(x^{6-n} + 3x^{10-n}\right)$$

$$= \begin{cases} 0 & \text{for } n < 6 \\ 1 & \text{for } n = 6 \\ \infty & \text{for } n > 6 \end{cases}$$

So, the largest n is $\boxed{6}$

b)