Lecture 6 Wed. Sept. 9th

Questions? Trouble understanding binomial formula

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{where} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

Stopping term

α, any real number

Note $\binom{\alpha}{0} = 0$

example: $\binom{10}{3} = \frac{10\cdot 9\cdot 8\cdot \cancel{7!}}{3!\,\cancel{7!}} = \frac{10\cdot 9\cdot 8}{3!}$ ← stopping term $(10-3+1)$

Probability

$\binom{n}{r}$ = the number of ways to choose r objects from n when order doesn't matter $= \frac{n!}{(n-r)!\,r!}$

$n, r \in$

Trouble understanding big O notation.

big O is a formal replacement to H.O.T. (higher order terms)

$$\cos x = 1 - \frac{x^2}{2!} + \text{H.O.T}$$

$$\cos x = 1 - \frac{x^2}{2!} + O(x^4)$$

term of order x^4 and higher

① For all x "near" 0, we say $f(x)$ is $O(g(x))$ as $x \to 0$ if $|f(x)| \le c\,|g(x)|$ for some constant C

" $\frac{f(x)}{g(x)}$ is bounded as $x \to 0$"

② For all large x, we say $f(x)$ is $O(g(x))$ as $x \to \infty$ if $|f(x)| \le c\,|g(x)|$ for some constant C

The definition of the **derivative**:

Derivative (first definition)

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

If the limit does not exist, then the derivative is not defined at a.

This first definition emphasizes that the derivative is the rate of change of the output with respect to the input. The next definition is similar.

Derivative (second definition)

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

If the limit does not exist, then the derivative is not defined at a.

This definition can be interpreted as the change in output divided by the change in input, as the change in input goes to 0. One can see this is equivalent to the first definition by making the substitution $h = x - a$. The third definition looks quite different from the first two.

Third definition

$$f\big(\underbrace{a+h}_{\text{Change in input by a small amount. "perturbation"}}\big) = f(a) + \underbrace{\left[\left.\frac{df}{dx}\right|_{x=a}\right]}_{\text{First order variation of the output}} \cdot h + \underbrace{O\left(h^2\right)}_{\to 0 \text{ as } h\to 0}$$

measures the corresponding change in the output at $x=a$

$$f(x+h) = f(x) + \underbrace{\frac{df}{dx}}_{\text{formula for the derivative at any value } x} h + O(h^2).$$

Derivative Notation

Best Notation

$$\frac{df}{dx}$$

$$\frac{dy}{dx}$$

Clear that x is the input and for y is the output

Fair Notation ← "just ok"

f' prime notation

$\dot{y}$ ← usually used when time t is the independent variable

dy ← differential

Examples

$x(t)$ = position as a function of time

$\frac{dx}{dt} = \frac{\text{change in position}}{\text{change in time}}$ = velocity = $v(t)$

$\frac{dv}{dt} = \frac{\text{change in velocity}}{\text{change in time}}$ = acceleration = $a(t) = \underbrace{\frac{d}{dt}\left(\frac{dx}{dt}\right)}_{\frac{d^2x}{dt^2}}$

the second deriv. of position as a function of time

$Q(t)$ = charge in a circuit as a function of time

$\frac{dQ}{dt} = \frac{\text{change in charge}}{\text{change in time}} = \underbrace{\text{current}}_{I(t)}$

Differentiation rules

Suppose u and v are differentiable functions of x. Then the following rules (written using the shorthand differential notation) hold:

LINEARITY

$$d(u+v) = du + dv \quad \text{and} \quad d(c \cdot u) = c \cdot du,$$ where c is a constant.

PRODUCT

$$d(u \cdot v) = u \cdot dv + v \cdot du.$$

CHAIN

$$d(u \circ v) = du \cdot dv.$$

For the quotient rule, you can think of it as a product rule.

$$h(x) = \frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)} = f(x) \cdot [g(x)]^{-1}$$

$$h'(x) = \underbrace{f'(x) \cdot [g(x)]^{-1}}_{\frac{f'(x)}{g(x)}} + \underbrace{f(x) \cdot -1[g(x)]^{-2} \cdot g'(x)}_{-\frac{f(x)g'(x)}{[g(x)]^2}} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Use the third definition of the derivative to prove the product rule

$$f(x+h) = f(x) + \frac{df}{dx}h + O(h^2).$$

$$f(x) = u(x) \cdot v(x) = (u \cdot v)(x)$$

$$f(x+h) = (u \cdot v)(x+h)$$

$$f(x+h) = \overbrace{u(x+h)} \cdot v(x+h)$$

$$f(x+h) = \left[u(x) + \frac{du}{dx} \cdot h + O(h^2)\right] \cdot \left[v(x) + \frac{dv}{dx} \cdot h + O(h^2)\right]$$

$$f(x+h) = u(x) \cdot v(x) + u(x)\frac{dv}{dx} \cdot h + u(x) \cdot O(h^2)$$
$$+ v(x)\frac{du}{dx}h + \frac{du}{dx}\frac{dv}{dx}h^2 + \frac{du}{dx} \cdot h \cdot O(h^2)$$
$$+ v(x)O(h^2) + \frac{dv}{dx}h\,O(h^2) + O(h^2)O(h^2)$$

$$f(x+h) = u(x) \cdot v(x) + \left[u(x)\frac{dv}{dx} + v(x) \cdot \frac{du}{dx}\right] \cdot h + O(h^2)$$

$$f(x+h) = u(x)\cdot v(x) + \left[u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}\right]\cdot h + O(h^2)$$

$$f(x+h) = f(x) + \underbrace{\left[u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}\right]}_{\text{the derivative of } u(x)\cdot v(x)}\cdot h + O(h^2)$$