DAY 8
Mon Sep 14

HIGHER DERIVATIVES & OPTIMIZATION

I. HIGHER DERIVATIVES

There's no reason to stop at "one" derivative, we can find higher derivatives to get a better (local) approximation to some complicated function!

Q1 If $f(x) = e^{-\sin(x)}$, find $f''(\pi)$.

Ans First, use the CHAIN RULE:

CHAIN
$\frac{d}{dx} u(v(x)) = u'(v(x)) \cdot v'(x)$

$$f'(x) = e^{-\sin x} \cdot \frac{d}{dx}(-\sin x)$$

So, $$f'(x) = e^{-\sin x} \cdot (-\cos x) \quad \text{- - - -} \quad (*)$$

Do NOT plug $x = \pi$ yet — We need to differentiate again! This time, use the PRODUCT RULE:

PRODUCT
$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)v(x) + v'(x)u(x)$

$$f''(x) = \frac{d}{dx}\left(e^{-\sin x}\right) \cdot (-\cos x) + \frac{d}{dx}(-\cos x) \cdot e^{-\sin x}$$

We ALREADY KNOW $\frac{d}{dx}\left(e^{-\sin x}\right)$ [see (*)], so don't re-compute it. Now,

$$f''(x) = \left[-e^{-\sin x}\cos^2 x + e^{-\sin x}\sin x\right]$$

$$= e^{-\sin x}\left[\sin x - \cos^2 x\right]$$

Plug in $x = \pi$ now: $\sin(\pi) = 0$, $\cos(\pi) = -1$:

$$f''(\pi) = e^0 \cdot \left[0 - (-1)^2\right] = \boxed{1}$$

The two essential things here are:

1. Do <u>not</u> plug x = (value) early, and
2. Do <u>not</u> re-compute derivatives you already have

Q2 If $f(x) = x^2 \ln x$, find $f^{(3)}(2)$

Ans This is <u>NOT</u> bad, even though we need <u>three</u> derivatives, look: by product rule,

$$f'(x) = (2x)\ln x + x^2 \cdot \left(\tfrac{1}{x}\right)$$
$$= 2x\ln x + x$$

Next,

more product rule...

$$f''(x) = 2\left[x\left(\tfrac{1}{x}\right) + (1)\ln x\right] + 1$$
$$= 2\left[1 + \ln x\right] + 1$$
$$= 3 + 2\ln x$$

Finally,

$$f^{(3)}(x) = \tfrac{2}{x},$$

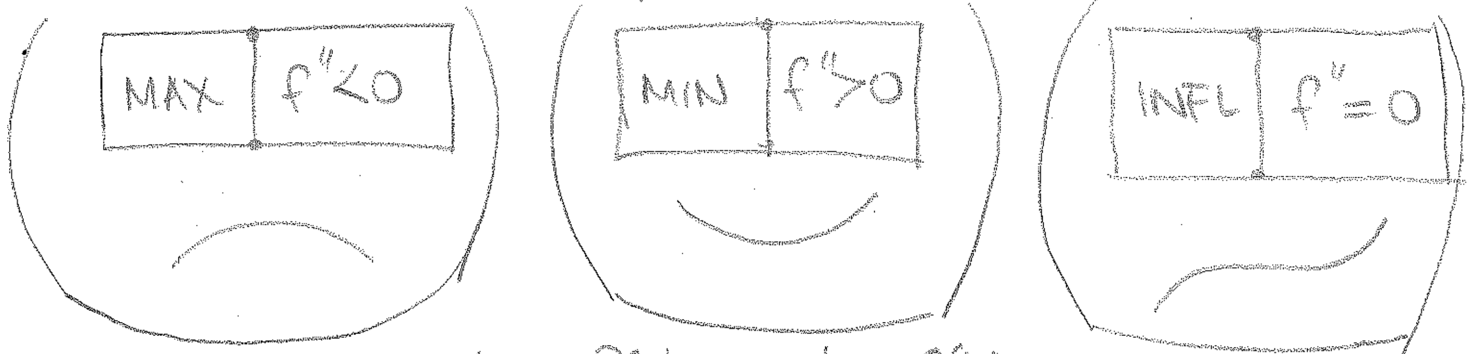
so $\boxed{f^{(3)}(2) = 1}$

OPTIMIZATION

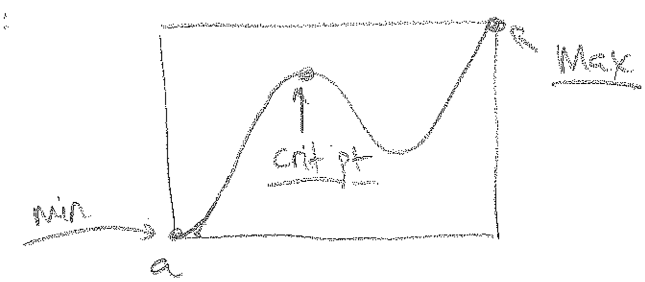
To find MAXIMA / MINIMA of $f(x)$ in $[a,b]$, we must:

- Find CRITICAL POINTS, where $f'(x) = 0$ or $f'(x)$ is undefined, and

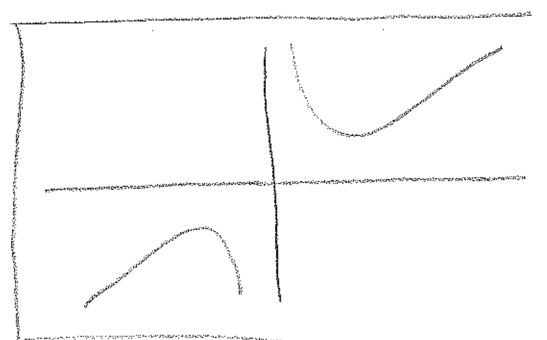
•• CLASSIFY critical points as



••• COMPARE with $f(a)$ and $f(b)$ to avoid situations like this:



Q3 For which x values is the sum $x + 1/x$ minimized?



Ans $f(x) = x + 1/x$, so

$f'(x) = 1 - 1/x^2$.

CRITICAL POINTS are solutions to $f'(x) = 0$, so $1/x^2 = 1$, so $\boxed{x = \pm 1}$

BUT note $f'(0)$ is undefined, so we must consider $\{-1, 0, 1\}$. Now,

$f(-1) = -2$, $f(1) = 2$, and

$f(0) = \ldots$ uh....

$f(x) = \frac{x^2+1}{x}$, so $\lim_{x \to \pm\infty} f(x) = \pm\infty$ By L'Hôpital,

Well, $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to 0^+} f(x) = +\infty$, so the function HAS NO MINIMUM!

Q4 Find the max and min of $f(x) = x^3 - 12x$ in the region $[0,5]$

Ans. Crit points are $f'(x) = 0$, so

$$3x^2 - 12 = 0$$

so, $x^2 = 4$,

so $x = \pm 2$.

Note $f''(x) = 4x$, so

✓ $+2$ is a MIN.

✗ -2 is a MAX $\longrightarrow$ But not in $[0,5]$, so discard!

Moreover, $f(2) = 8 - 24 = \underline{-16}$

We're NOT DONE: check f at endpoints:

$f(0) = -12$ $\longleftarrow$ not lower than $f(+2)$, ok ...

$f(5) = 125 - 60 = \underline{65}$, $\longleftarrow$ larger than $f(-2)$!

So, <u>MIN</u> is at $\boxed{x = 2}$, value $\underline{-16}$.

<u>MAX</u> is at $\boxed{x = 5}$, value $\underline{65}$.