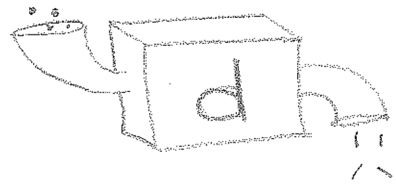
DAY 9
Wed Sep 16

DIFFERENTIALS



The Problem: Sometimes, we either don't get a "nice" $f(x) = \langle blah \rangle$, but rather a more complicated expression where f and x are hard / impossible to separate. Or maybe we don't know the derivative of the right side.

I	II	III
$f(x) = \arccos(x^2)$ $df/dx = ?$	$\sin(3x+y) = e^{-x^2 y}$ $dy/dx = ?$	$3(x-2)^2 + 6(y-1)^2 = 1$ What are the critical points of y as a function of x?

I. Set $y = \arccos(x^2)$, want dy/dx.

So, $\cos(y) = x^2$

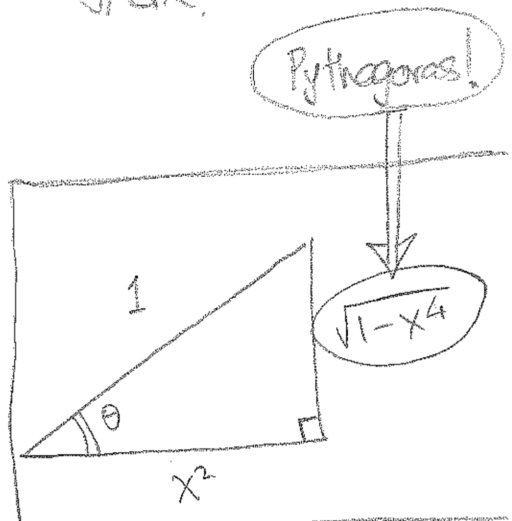
Hit both sides with "d":

$$d\cos y = d(x^2)$$

So, $-\sin y \, dy = 2x \, dx$

So, $$\frac{dy}{dx} = \frac{2x}{-\sin(y)}$$

or, $$\frac{dy}{dx} = \frac{-2x}{\sin(\arccos(x^2))}$$



$\sin\theta = \sqrt{1-x^2}$

Done? **No** WTF is $\sin(\arccos(x^2))$??

So, $\boxed{\dfrac{dy}{dx} = \dfrac{-2x}{\sqrt{1-x^4}}}$

II. $\sin(3x+y) = e^{-x^2y}$, find dy/dx.

It is IMPOSSIBLE here to get y as a function of x. But still, use d:

$$d\left[\sin(3x+y)\right] = d\left[e^{-x^2y}\right]$$

chain rule on both sides → So, $\cos(3x+y)\cdot d(3x+y) = e^{-x^2y}\, d(-x^2y)$

Product rule on Right side → So, $\cos(3x+y)\,(3dx+dy) = -e^{-x^2y}\cdot(2xy\,dx + x^2dy)$

Collect dx's to one side and dy's to other:

$$\left[3\cos(3x+y) - 2xye^{-x^2y}\right]dx = \left[-x^2e^{-x^2y} - \cos(3x+y)\right]dy$$

So, $\dfrac{dy}{dx} = \dfrac{\text{here}}{\text{here.}}$

NOTE: The derivative is in terms of BOTH y and x. This is <u>unavoidable</u> sometimes!

III. Critical points when $3(x-2)^2 + 6(y-1)^2 = 1$.

Here, we "could" solve for y, but there would be ugly square-roots, etc. So,

$$d\left[3(x-2)^2 + 6(y-1)^2\right] = d(1)$$

So, $6(x-2)\,dx + 12(y-1)\,dy = 0$

So, $\dfrac{dy}{dx} = \dfrac{-6(x-2)}{12(y-1)} = \dfrac{-(x-2)}{2(y-1)}$

Critical points occur when: $dy/dx = 0$ or undefined,
so either at $\boxed{x=2}$ or when $\underline{y=1}$. But
when does $y=1$? Go back to original equation:

$$3(x-2)^2 + 6(y-1)^2 = 1$$

Plug $y=1$, get $3(x-2)^2 = 1$,

So $(x-2)^2 = 1/3$

So $x-2 = \pm 1/\sqrt{3}$

So $\boxed{x = 2 \pm 1/\sqrt{3}.}$

THREE critical points: $\{2 - 1/\sqrt{3},\ 0,\ 2 + 1/\sqrt{3}\}$

LOGARITHMIC DIFFERENTIATION

When facing $y = f(x)^{g(x)}$ for ugly f and g, first compute $\ln$ of both sides and then use d. Eg:

IV. $y = (4x)^{8x}$, compute dy/dx.

$\ln y = 8x \ln(4x)$

So, $d[\ln y] = d[8x \ln(4x)]$

Chain rule on left, product rule on right...

$$\frac{dy}{y} = \left[8 \ln(4x)\, dx + 8x \left(\frac{1}{\cancel{4}x}\right) \cancel{4}\, dx \right]$$

$$= 8 \left[\ln(4x) + 1 \right] dx$$

So, $$\frac{dy}{dx} = 8y \left[\ln(4x) + 1 \right]$$

$$= 8 \cdot (4x)^{8x} \left[\ln(4x + 1) \right]$$

RELATIVE RATE OF CHANGE

This is JUST "dy/y", a normalization!

V. After time t with initial investment P_0, the return $P(t)$ is $P_0(1+r)^t$ where r = interest rate. What is the rel rate of change of P?

$\frac{d}{dt} a^t = a^t \ln a$

↑ check this!

$$dP = d\left[P_0 (1+r)^t \right]$$

$$= P_0 (1+r)^t \ln(1+r)$$

So, $$\frac{dP}{P} = \frac{\cancel{P_0 (1+r)^t} \ln(1+r)}{\cancel{P_0 (1+r)^t}}$$

$$= \boxed{\ln(1+r)}$$