DAY 11
Wed Sep 23

ANTI$\int$DIFFERENTIATION & ODEs I

(To Joy?)

An ANTIDERIVATIVE $\int f(x)\,dx$ of the function $f(x)$ is any function $F(x)$ so that $\boxed{dF/dx = f(x)}$!

Antiderivatives (unlike derivatives) are NOT UNIQUE, because $d\,\text{const.}/dx = 0$. So, if $F(x)$ and $G(x)$ are both antiderivatives of the same $f(x)$, then: $\boxed{F(x) - G(x) = \text{constant}}$!

Q1 What's the ANTIDERIVATIVE of $\sin(3x)$?

Ans Note $\frac{d}{dx}\cos(3x) = -3\sin(3x)$,

so, $\frac{d}{dx}\left[-\frac{1}{3}\cos(3x)\right] = \sin(3x)$

so, $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$

Q2 What's the antiderivative of x^n?

Ans. We "almost" have $\frac{d}{dx}x^{n+1} = (n+1)x^n$, the ONLY exception occurs at $n = -1$, where

$$\frac{d}{dx}\ln x = x^{-1}.$$

So: for $n \neq -1$, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

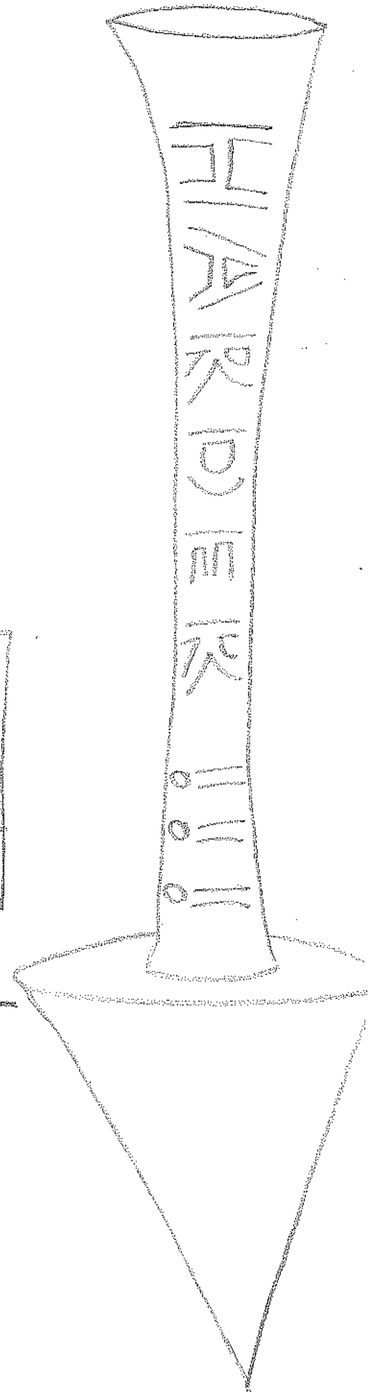
for $n = -1$, $\int x^{-1}\,dx = \ln(x) + C$

MOST IMPORTANT

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

ORDINARY DIFFERENTIAL EQUATIONS
(A BROAD VIEW...)

These are equations relating functions with their derivatives. We will only look at SIMPLE ones for now... treat x as a function of t:



$$dx/dt = a$$
(a constant)

$$x(t) = at + C$$


"Hahaha"

$$\frac{dx}{dt} = g(t)$$

$$x(t) = \int g(t)\,dt$$


"antiderivative problem"


"Autonomous"

$$dx/dt = f(x)$$
HARD! When $f(x)$ is just ax, then:

$$x(t) = Ce^{at}$$

IMPORTANT


"Separable"

$$dx/dt = f(x)g(t)$$
Also Hard!

$$x(t) = ?$$


"Aaaah..."

$$dx/dt = h(x,t)$$
h may not "split"

OMFG!!

Q3. If a population (with no new births) has a 2% mortality rate, how long will it take until only 10% of the people remain alive?

Ans

$$dP/dt = -aP(t) \qquad \left[\begin{array}{l}\text{where} \\ a = 2\% = 0.02\end{array}\right]$$

So, $P(t) = P(0)\, e^{-at}$

Want: to find t-value where $P(t) = \frac{P(0)}{10}$, so let's solve:

$$\frac{\cancel{P(0)}}{10} = \cancel{P(0)}\, e^{-at}$$

$$\text{So,} \quad e^{-at} = 1/10$$

$$-at = \ln(1/10)$$

$$t = -1/a \, \ln(1/10)$$

[For $a = 0.02$, $t \approx \underline{\underline{115.13}}$ years.]

Q4. Which function $x(t)$ satisfies $dx/dt = x^2\cos(t)$ and $x(0) = 1$ | Separable!

Ans

$$\frac{dx}{dt} = x^2\cos(t)$$

So, $$\frac{dx}{x^2} = \cos(t)\,dt$$

$$\Rightarrow \int \frac{dx}{x^2} = \int \cos(t)\,dt$$

So, $\dfrac{x^{-2+1}}{-2+1} = \sin(t) + C$

$\Rightarrow$ $\boxed{-1/x = \sin(t) + C}$

Just need to figure out C ... Well, at $t=0$, $x=1$. So,

$$-1/1 = \sin(0) + C,$$

so $C = -1$. Therefore,

$$-1/x = \sin(t) - 1$$

or, $x = \dfrac{-1}{\sin(t) - 1}$

$$= \boxed{x = \frac{1}{1 - \sin(t)}}$$

By the way, you can ALWAYS check your answers to diff eq problems.

$$d\left[\ x = \frac{1}{1-\sin t}\ \right]$$

Get: $dx = \cancel{-}(1-\sin t)^{-2} \cdot (\cancel{-}\cos t)\, dt$

So, $\dfrac{dx}{dt} = x^2 \cos t$ ✓ 16!