DAY 12
Mon Sep 28

$\int$NTEGRATING FACTORS, EQUILIBRIA & LINEARIZATION of ODEs

RECAP: Separable ODE

Sometimes we need algebra to separate $dx/dt = f(x)\,g(t)$

Q1 Solve $\dfrac{dx}{dt} = xt^2 + x - t^2 - 1$, $x(0) = 2$

Ans. Note: $\dfrac{dx}{dt} = x\cdot(t^2+1) - 1\cdot(t^2+1)$ ← Factor...

$= (x-1)(t^2+1)$ ← Separable!

So, $\dfrac{dx}{x-1} = (t^2+1)\,dt$.

Integrate: $\displaystyle\int \frac{dx}{x-1} = \int (t^2+1)\,dt$

$\Rightarrow \ln(x-1) = t^3/3 + t + C$

$\Rightarrow x - 1 = e^{t^3/3 + t + C}$

When $t = 0$, $x = 2$, so:

$1 = e^C$, so $C = 0$.

Finally, $\boxed{x(t) = e^{t^3/3 + t} + 1}$

LINEAR ODE:

These are NOT separable (in the worst case) but have the form

$$\frac{dx}{dt} = \underbrace{A(t)\,x + B(t)}_{\text{Right side "linear" in } x\text{-variable only.}}$$

To solve, set $I(t) = e^{-\int A(t)\,dt}$, then:

Comes from Product rule, $d(I(t)\,x(t))$ etc. $\rightarrow$ $\boxed{x(t) = \frac{1}{I(t)} \int I(t) B(t)\, dt}$

Q2

$$\frac{dx}{dt} = \frac{1}{t}\left[x + t^4\right].$$

Not separable! But...

$$\frac{dx}{dt} = \frac{x}{t} + t^3$$

$\leftarrow$ Linear! with $A(t) = 1/t$ and $B(t) = t^3$.

$$\text{So, } I(t) = e^{\int -1/t\, dt}$$
$$= e^{-\ln(t)}$$
$$= e^{\ln(t^{-1})}$$
$$= t^{-1}$$

$$\text{Finally } x(t) = \frac{1}{I(t)} \int I(t) B(t)\, dt$$
$$= t \int t^{-1} \cdot t^3\, dt$$
$$= t \int t^2\, dt$$
$$= t \cdot \left(t^3/3 + C\right)$$
$$= \boxed{t^4/3 + Ct}$$

EQUILIBRIA: For autonomous equations, i.e., $dx/dt = f(x)$, the equilibria are those x where $\boxed{f(x) = 0}$

And... each equilibrium is usually "stable" or "unstable" depending on the sign of f'. (is this familiar?)

Q3 Find and classify all equilibria of

$$\boxed{\frac{dx}{dt} = x^3 - 3x^2 + 2x}$$

Ans.

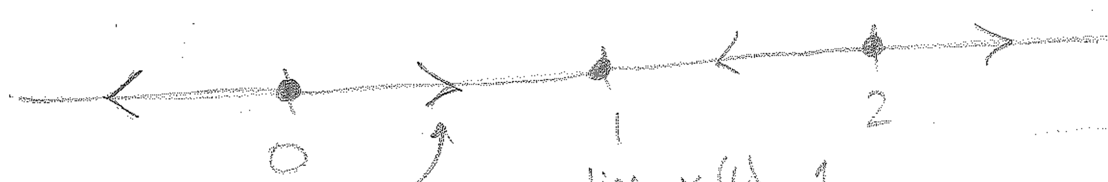
$$f(x) = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2)$$
$$= x(x-1)(x-2).$$

So equilibria are $\underline{x=0}$ $\underline{x=1}$ and $\underline{x=2}$.

$f'(x) = 3x^2 - 6x + 2$, which is:

- $= 2 > 0$ at $x = 0$, [unstable]
- $= -1 < 0$ at $x = 1$, [stable]
- $= 2 > 0$ at $x = 2$ [unstable]

PICTURE:



So, if $x(0) = \frac{1}{2}$, $\lim_{t\to\infty} x(t) = 1$.
$\lim_{t\to-\infty} x(t) = 0$.

We can predict LONG-TERM behavior WITHOUT solving!

Q4 Given

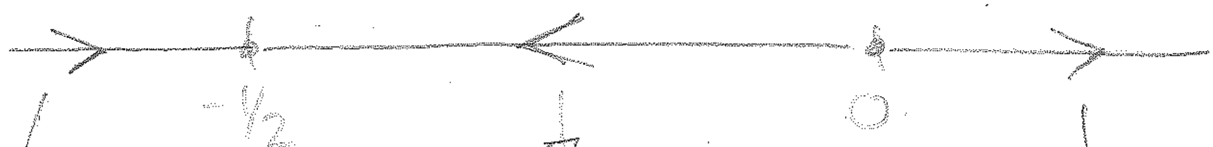
$$\frac{dx}{dt} = (e^x - 1)(1 + 2x) \quad \text{and} \quad x(0) = -0.1$$

Find $\lim_{t \to \infty} x(t)$ and $\lim_{t \to -\infty} x(t)$

Ans $f(x) = (e^x - 1)(1 + 2x)$,

This equals 0 at $x = 0$ and $x = -\frac{1}{2}$

Now, $f'(0)$ and $f'(-\frac{1}{2})$ are hard to compute (product rule, etc) so let's just check signs of $f(x)$ at intermediate points.

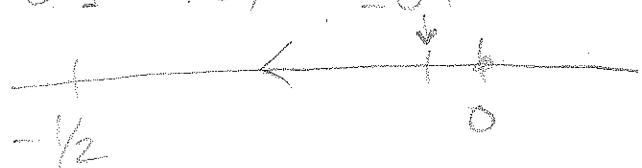


At $x = -1$, $e^x - 1$ and $1+2x$ are both negative, so $f > 0$ and x is increasing.

At $x = -\frac{1}{4}$, $e^{-1/4} - 1 < 0$ and $1 - 2/4 > 0$, so $f < 0$ and x is decreasing

At $x = 1$, $e^x - 1 > 0$, $1 + 2x > 0$, so $f > 0$ and x is increasing

Here is $-0.1 = x(0)$: -0.1



Clearly, $\lim_{t \to \infty} x(t) = -\frac{1}{2}$ and $\lim_{t \to -\infty} x(t) = 0$.