DAY 13
WED SEP 30

SUBSTITUTION, INTEGRATION BY (PAR)(TS)

Today, we learn two SUPER-IMPORTANT techniques to solve integrals that are not "obvious" antiderivatives, things more complicated than $\{x^n, \cos, \sin, e^{\circ}, \ldots\}$

SUBSTITUTION : $\int f(u(x)) \cdot u'(x)\,dx = \int f(u)\,du$, the Anti-Chain Rule!

Q1 A) Solve $\int \cos^2 x \sin x\,dx$.
B) And $\int \cos^2 x \sin^3 x\,dx$.

Ans. A) : Set $u = \cos x$, so $du = -\sin x\,dx$

Then, $\int \cos^2 x \sin x\,dx = \int u^2 \cdot (-du)$

$$= -u^3/3 + C = \boxed{-\cos^3 x/3 + C}$$

B). Start again with $u = \cos(x)$, so $du = -\sin x$. But now we have "$\sin^3 x\,dx$", not what we want! But look...

$$\int \cos^2 x \sin^3 x\,dx = \int \cos^2 x \sin^2 x \cdot \sin x\,dx$$

$$= -\int \cos^2 x\,(1-\cos^2 x)\,\sin x\,dx \quad \leftarrow \text{(Ha!)}$$

$$= -\int u^2(1-u^2)\,du = \text{(easy now...)}$$

Q2 $\int x\sqrt{3-x}\,dx = ?$

Chase's Law: $\sqrt{3-x} \neq \sqrt{3}-\sqrt{x}$.

Ans The problem is $\sqrt{3-x}$, so let's substitute it away! Set

$u = 3-x$, so $du = -dx$

And, $x = 3-u$

Now, $\int x\sqrt{3-x}\,dx = -\int (3-u)\,u^{1/2}\,du$

$= -\int (3u^{1/2} - u^{3/2})\,du$

$= -\frac{3u^{1/2+1}}{1/2+1} + \frac{u^{3/2+1}}{3/2+1} + C$

$= -2u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= -2(3-x)^{3/2} + \frac{2}{5}(3-x)^{5/2} + C$

Q3. $\int \frac{[3-\ln^2(x)][2-\ln^3(x)][1+\ln^5(x)]}{7x}\,dx = ?$

Ans. Set $u = \ln(x)$, $du = dx/x$ to get

$\int \frac{(3-u^2)(2-u^3)(1+u^5)}{7}\,du$, now solve!

Q4. $\int \sqrt{1-\sqrt{x}}\,dx = ?$

Ans. This takes a BIT MORE...

Set $u = 1-\sqrt{x}$, so $du = -\frac{1}{2\sqrt{x}}\,dx$.

But our integrand does not have $-\frac{1}{2\sqrt{x}}\,dx$

Sometimes called "Back-substitution"

In this case, TRY expressing dx in terms of du, like this:

$$-\frac{1}{2\sqrt{x}}\,dx = du$$

$$\text{so} \quad dx = -2\sqrt{x}\,du$$

Note $u = 1-\sqrt{x}$, so $\sqrt{x} = 1-u$ which gives

$$dx = -2(1-u)\,du = \underline{2(u-1)\,du}$$

Finally, $\displaystyle\int \underbrace{\sqrt{1-\sqrt{x}}}_{u}\,\underbrace{dx}_{2(u-1)} = \int u^{1/2}\cdot 2(u-1)\,du$

(Now easy!)

INTEGRATION BY PARTS: the ANTI-PRODUCT RULE

$$\underbrace{\int u\,dv}_{\text{(Hard)}} = uv - \underbrace{\int v\,du}_{\text{(easy)}},$$

Q5

$$\int 2x\cos x\,dx = \,?$$

Ans

Integrating/differentiating $\cos(x)$ doesn't make it any simpler. But the derivative of $2x$ is much nicer than its antiderivative. So,

set $u = 2x$, $\quad dv = \cos x\,dx$

so $du = 2dx$, $\quad v = \sin x$.

Now, $\displaystyle\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx$

$$= \boxed{2x\sin x + 2\cos x + C}$$

See? That was EASY...

Q6 | Try $\int x^2 \sin(2x)\,dx$

Ans. Set $u = x^2$, $dV = \sin(2x)\,dx$

so, $du = 2x\,dx$, $V = -2\cos(2x)$

Now, $\int x^2 \sin(2x)\,dx = -2x^2\cos(2x) + \underbrace{\int 4x\cos(2x)}_{\text{this}}$

Use Q5 - type method to solve this)

Q7 | Solve $I = \int e^{-x} \sin x\,dx$. What is $\underline{I}$?

$u = e^{-x}$ $dv = \sin x\,dx$

$du = -e^{-x}dx$ $V = -\cos x$

So, $I = -e^{-x}\cos x - \underbrace{\int e^{-x}\cos x\,dx}_{\text{call this "J"}}$ $(\ast)$

$J = \int e^{-x}\cos x\,dx$. Again, by parts!

$u = e^{-x}$, $dv = \cos x\,dx$

$du = -e^{-x}dx$, $V = \sin x$

So, $J = e^{-x}\sin x + \int e^{-x}\sin x\,dx$

$= e^{-x}\sin x + I$

Use this in $(\ast)$ above to get

$I = -e^{-x}\cos x - (e^{-x}\sin x + I)$

So, $2I = -e^{-x}(\cos x + \sin x)$, so

$$\boxed{I = -\frac{e^{-x}}{2}(\cos x + \sin x) + C}$$

Q8 | What is $\int \frac{\ln^2 x}{x^2}\,dx$?

Ans | Set $y = \ln(x)$, so $dy = \frac{dx}{x}$ and $x = e^y$.

Now,

$$\int \frac{\ln^2(x)}{x^2}dx = \int \frac{\ln^2(x)}{x}\,\frac{dx}{dx}$$

$$= \int \frac{y^2}{e^y}\,dy = \int y^2 e^{-y}\,dy = J.$$

"J" we can solve by parts (twice). In class we used the "Table" method, i.e.

Thanks, Allie...!

derivatives of y^2		antiderivatives of e^{-y}
y^2		e^{-y}
$2y$	$+$	$-e^{-y}$
2	$-$	$+e^{-y}$
0	$+$	$-e^{-y}$

Sum up diagonals with alternating signs...

So, $J = -(y^2e^{-y} + 2ye^{-y} + 2e^{-y})$

$= -(y^2+2y+2)e^{-y}$

Plug in $y = \ln x$, to get

Ans $= \boxed{-(\ln^2(x) + 2\ln(x) + 2)\cdot\frac{1}{x}}$