DAY 14
Fri Oct 2

TRIG-SUB & PARTIAL FRACTIONS

If you see...	Try...	Because...
$\sqrt{1+x^2}$	$x = \tan\theta$	$\sqrt{1+\tan^2\theta} = \sec\theta$
$\sqrt{1-x^2}$	$x = \sin\theta$ (or $\cos\theta$)	$\sqrt{1-\sin^2\theta} = \cos\theta$
$\sqrt{x^2-1}$	$x = \sec\theta$ (or $\csc\theta$)	$\sqrt{\sec^2\theta - 1} = \tan\theta$

Q1. $\int \sqrt{3-x^2}\,dx$

Ans. Want: $\sqrt{3-x^2} = \sqrt{3(1-\sin^2\theta)} = \sqrt{3}\cos\theta$, so

try: $x = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta\,d\theta$

[also note: $\theta = \arcsin(x/\sqrt{3})$].

Now,

$$\int \sqrt{3-x^2}\,dx = \int \sqrt{3}\cos\theta \cdot \sqrt{3}\cos\theta\,d\theta$$

$$= 3\int \cos^2\theta\,d\theta.$$

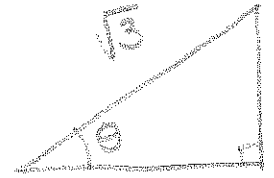
Need: $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

$$= \tfrac{3}{2}\int [1+\cos(2\theta)]\,d\theta$$

$$= \tfrac{3\theta}{2} + \tfrac{3}{4}\sin(2\theta) + C.$$

But: what is $\sin(2\theta)$ in terms of x?

Well, $\theta = \arcsin\left(x/\sqrt{3}\right)$, so it looks like:



This must be $\sqrt{3-x^2}$ by Pythagoras

So, $\sin(\theta) = x/\sqrt{3}$ (obvious) —— $(*)$

and $\cos(\theta) = \dfrac{\sqrt{3-x^2}}{\sqrt{3}}$ (less obvious) —— $(*)$

Now, $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$, so coming back to our expression:

$$\int \sqrt{3-x^2}\,dx = \frac{3\theta}{2} + \frac{3}{4}\sin(2\theta) + C$$

$$= \frac{3\theta}{2} + \frac{3}{4}\cdot 2\sin\theta\cos\theta + C.$$

[Use the $(*)$'s above.]

$$= 3\arcsin\left(x/\sqrt{3}\right) + \frac{3}{2}\,\frac{x\sqrt{3-x^2}}{\sqrt{3}\cdot\sqrt{3}} + C$$

$$= \boxed{3\arcsin\left(x/\sqrt{3}\right) + \frac{x\sqrt{3-x^2}}{2} + C}$$

Q2 $\displaystyle\int \frac{x\,dx}{\sqrt{2x^2+4}} = ?$

Ans. Want $\sqrt{2x^2+4} = \sqrt{4(\tan^2\theta+1)} = 2\sec\theta$, so use $2x^2 = 4\tan^2\theta$, or $x = \sqrt{2}\tan\theta$.

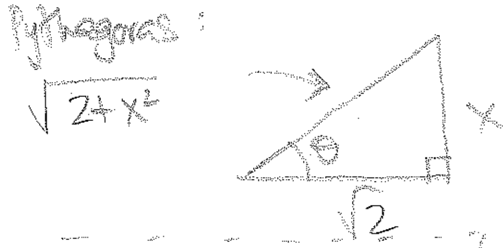
Then, $dx = \sqrt{2}\sec^2\theta\,d\theta$ and $\theta = \arctan\left(x/\sqrt{2}\right)$

So,

$$\int \frac{x\,dx}{\sqrt{2x^2+4}} = \int \frac{2\tan\theta}{2\sec\theta} \cdot \sqrt{2}\sec^2\theta \cdot d\theta$$

$$= \sqrt{2}\int \tan\theta\sec\theta\,d\theta = \sqrt{2}\sec\theta + C.$$

To get $\sec\theta$ in terms of x, use $\theta = \arctan(x/\sqrt{2})$.



So, $\boxed{\sec\theta = \dfrac{\sqrt{2+x^2}}{\sqrt{2}}}$. And finally,

$$\int \frac{x\,dx}{\sqrt{2x^2+4}} = \sqrt{2}\sec\theta + C$$

$$= \boxed{\sqrt{2+x^2} + C}$$

Q3 $$\int \frac{x\,dx}{\sqrt{x^2-4x+7}}$$

Ans. First, get the denominator in friendly form: ("complete the square"):

$$x^2-4x+7 = (x-2)^2+3, \text{ so want:}$$

$$(x-2)^2+3 = 3\tan^2\theta + 3 = 3(\sec^2\theta).$$

So, try: $x-2 = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$

and $\theta = \arctan\left(\dfrac{x-2}{\sqrt{3}}\right)$

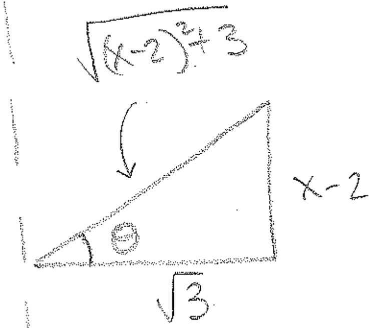
Now,

$$\int \frac{x\,dx}{\sqrt{x^2-4x+7}} = \int \left(\frac{2+\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\right) \cdot \sqrt{3}\sec^2\theta\,d\theta$$

$$= \int 2\sec\theta + \sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$= 2\ln|\sec\theta + \tan\theta| + \sqrt{3}\sec\theta\,d\theta + C$$

Again, to get this in terms of x



$$\sec\theta = \frac{\sqrt{(x-2)^2+3}}{\sqrt{3}}$$

$$\tan\theta = \frac{x-2}{\sqrt{3}}$$

So,

$$\int \frac{x\,dx}{\sqrt{x^2-4x+7}} = \boxed{2\ln\left|\frac{\sqrt{(x-2)^2+3}}{\sqrt{3}} + \frac{x-2}{\sqrt{3}}\right| + \sqrt{(x-2)^2+3} + C}$$

PARTIAL FRACTIONS

Q4

$$\int \frac{x+5}{2x^2+x-3}\,dx = ?$$

Ans. First: factor denominator.

$$(2x^2+x-3) = (2x+3)(x-1)$$

Next, solve for A and B in:

$$\frac{x+5}{(2x+3)(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}, \text{ like so.}$$

First, cross-multiply by denominator on LEFT

$$x+5 = A(x-1) + B(2x+3).$$

Now, plug in "good" values of x designed to zero-out things on the RIGHT:

@ $x=1$, get $6 = 5B$, so $B = 6/5$
@ $x = -3/2$, get $7/2 = A(-3/2)$, so $A = -7/3$

Finally, $$\frac{x+5}{(2x+3)(x-1)} = -7/3 \, \frac{1}{2x+3} + 6/5 \, \frac{1}{x-1}.$$

Now integrate:

$$\int \frac{x+5}{2x^2+x-3}\,dx = -7/3 \int \frac{dx}{2x+3} + 6/5 \int \frac{dx}{x-1}$$

$$= \boxed{-7/3 \, \frac{\ln|2x+3|}{2} + 6/5 \ln|x-1| + C}$$

Q5 $$\int \frac{dx}{x^2+3x+2} = ?$$

Again, start by factoring (x^2+3x+2)

$$x^2+3x+2 = (x+2)(x+1)$$

Now, $\dfrac{1}{x^2+3x+2} = \dfrac{A}{x+2} + \dfrac{B}{x+1}$

Or, $1 = A(x+1) + B(x+2)$

@ $x=-1$, $1 = B$.

@ $x=-2$, $1 = -A$.

So, $\dfrac{1}{x^2+3x+2} = \dfrac{-1}{x+2} + \dfrac{1}{x+1}$

Integrate:

$$\int \frac{dx}{x^2+3x+2} = -\ln|x+2| + \ln|x+1| + C.$$

$$= \ln\left|\frac{x+1}{x+2}\right| + C.$$