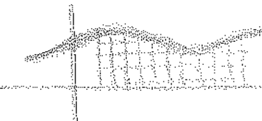
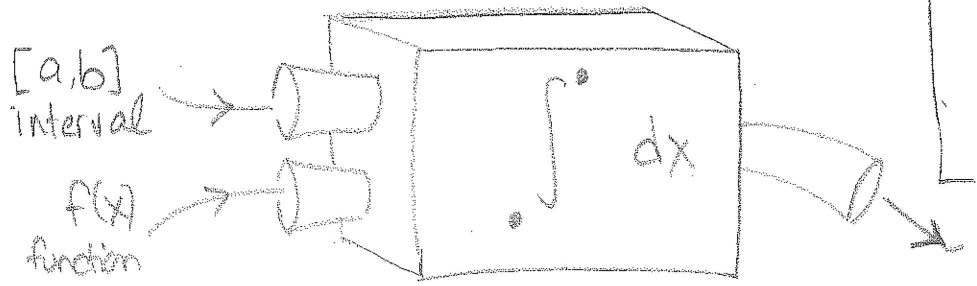
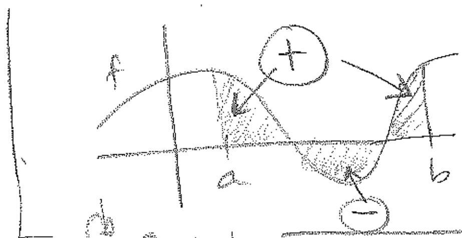
DAY 15
MON OCT 5

THE FUNDAMENTAL THEOREM



- Defining the DEFINITE INTEGRAL: $\int_a^b f(x)\,dx$



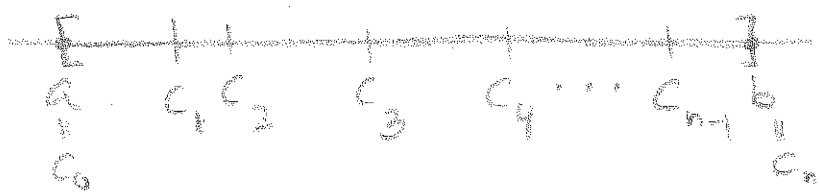


$\int_a^b f(x)\,dx$, NUMBER!

[a,b] "in" domain of f.

PARTITION

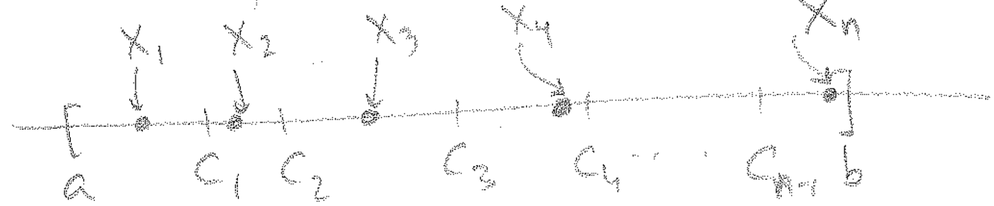
Given [a,b], first CHOP it into "n" pieces, by choosing $a = c_0 < c_1 < \cdots < c_{n-1} < c_n = b$.



- let $P = \{[c_i, c_{i+1}]\}_0^{n-1}$,
- $|P| = \max\limits_{0 \le i \le n-1} [c_{i+1} - c_i]$ ← maximum width

SAMPLE

Now, choose ANY $x_1, x_2, \dots, x_n$ so that $x_\bullet$ is in $[c_{\bullet-1}, c_\bullet]$: let $\mathbb{X} = \{x_1, \dots, x_n\}$



SUM

The RIEMANN SUM for this partition P, this sample $\mathbb{X}$

$$\boxed{R_P(\mathbb{X}) = \sum_{i=0}^{n} f(x_i) \cdot [c_{i+1} - c_i]}$$

<u>Def</u> $\boxed{\int_a^b f(x)\,dx = \lim_{|P|\to 0} R_P(\Xi)}$

One can use this definition to compute "signed" area only in the <u>SIMPLEST</u> of cases: $f = \text{const}$, $f = ax+b$, and maybe a little more.

THE FUNDAMENTAL THEOREM (OF INTEGRAL CALCULUS)

It has MANY faces:

✓ I. $\int_a^b f'(x)\,dx = f(b) - f(a)$

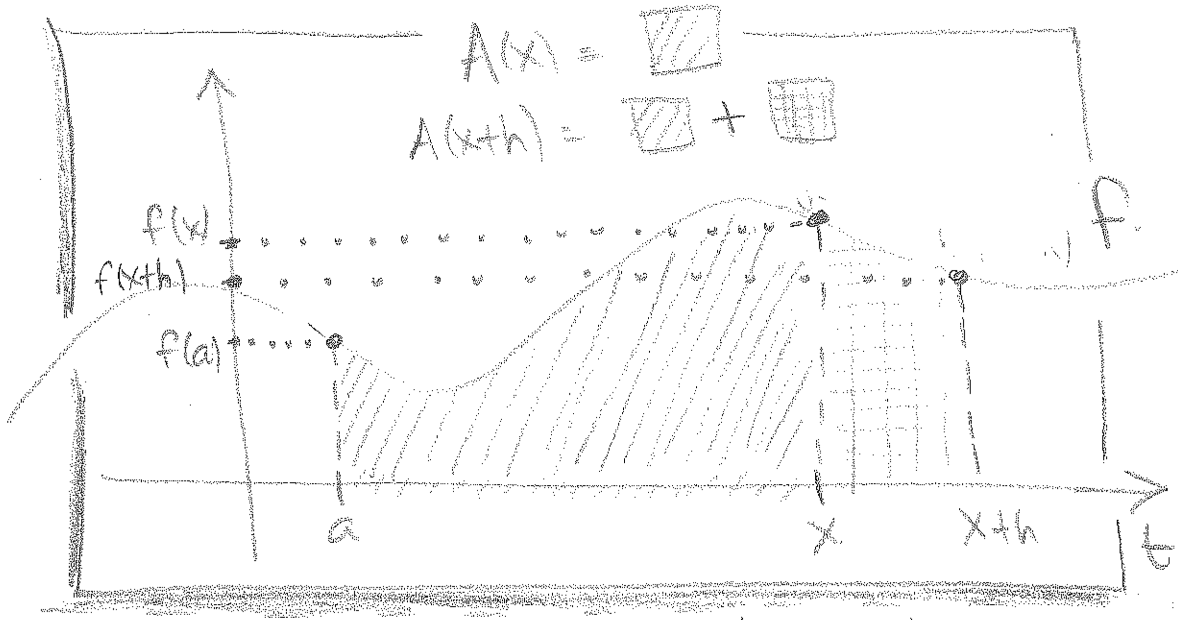
✓ II. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

X. [As a consequence, ... $\sum_{i=1}^{n} (a_{i+1} - a_i) = a(n) - a(1)$ for any sequence $a(1), a(2), \ldots, a(n)$ of numbers]

<u>WHY</u> should this be true?? Let's focus on:

II. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

For each $x \geq a$ let $A(x)$ be the (signed) area under the graph of f between a and x



We will compute $\frac{d}{dx} A(x)$ by definition:

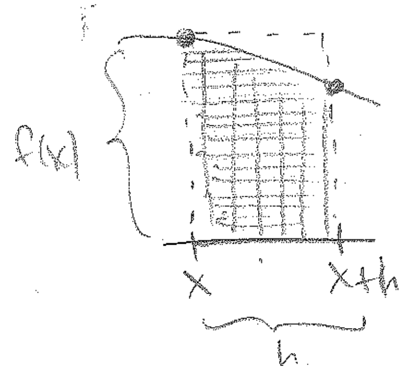
i.e., $\displaystyle\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$.

Now, $\displaystyle A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$

This is just: $\displaystyle\int_x^{x+h} f(t)\,dt$

So, $\displaystyle\frac{A(x+h) - A(x)}{h} = \frac{1}{h} \underbrace{\int_x^{x+h} f(t)\,dt}$

(Roughly a rectangle of area) $\underline{f(x) \cdot h}$.



So, $\displaystyle\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$

Upshot: We can use FTC to calculate Definite Integrals!

1. $$\int_3^5 x^3\,dx = ?$$

2. $$\int_{-\pi}^{\pi} \sin^2\theta\,d\theta = ?$$

3 $$\int_{-1}^{3} dx/x^3$$

CAREFUL! The domain of $1/x^3$ does NOT include 0!

4 $$\int_0^1 e^{-x}\,dx = ?$$