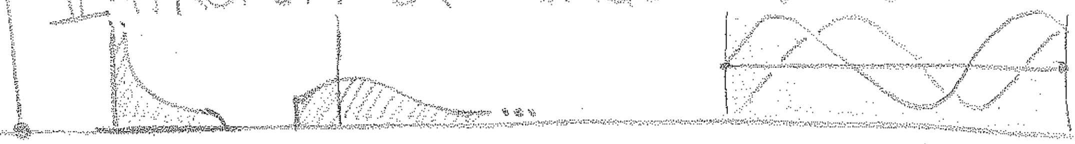
DAY 16
WED OCT 7

IMPROPER & TRIGONOMETRIC INTEGRALS



RECAP: FUNDAMENTAL THEOREM, if $dF/dx = f(x)$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Assumes that $[a,b]$ is in the domain of f.

An $\boxed{\text{IMPROPER INTEGRAL}}$ is one where

- some points in $[a,b]$ are not in the domain, OR
- either $a = -\infty$ or $b = +\infty$ or both.

This means that FTC fails "directly", but there may still be hope to solve such integrals...

WARNING EXAMPLE

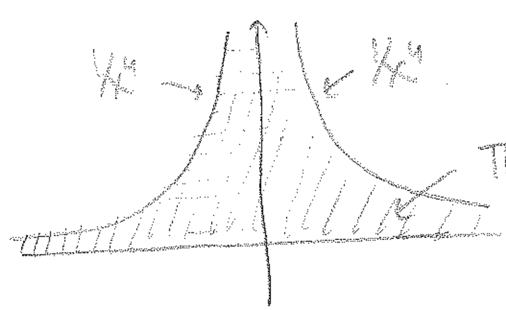
$$I = \int_{-1}^{1} x^{-4}\,dx$$

If you try using the FTC, then... $\int x^{-4}\,dx = x^{-3}/{-3}$.

$$I = x^{-3}/{-3}\Big|_{x=-1}^{x=1}$$

$$= 1/{-3} - (-1)/{-3} = -2/3$$

But:



This is not "negative" area, $1/x^4 \geq 0$

The way out requires breaking up the integral into pieces (depending on where the singularities are)...

Q1 | What is $\int_{-1}^{1} x^{-4}\,dx$?

The troublesome point is 0, so:

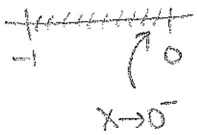
$$\int_{-1}^{1} x^{-4}\,dx = \int_{-1}^{0} x^{-4}\,dx + \int_{0}^{1} x^{-4}\,dx$$

- Deal with the first integral:

$$\int_{-1}^{0} x^{-4}\,dx = x^{-3}/{-3}\Big|_{x=-1}^{x=0}$$

Plugging in zero directly does not make sense..
Instead, use LIMITS.

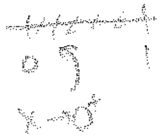
$$x^{-3}/{-3}\Big|_{x=-1}^{x=0} = \left[\lim_{x\to 0^-} x^{-3}/{-3}\right] - (-1)^3/{-3}$$



But this stuff is already $+\infty$.

- Now, the second integral:

$$\int_{0}^{1} x^{-4}\,dx = x^{-3}/{-3}\Big|_{x=0}^{x=1}$$



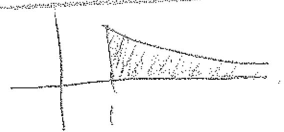
$$= (1)^3/{-3} - \lim_{x\to 0^+} x^{-3}/3 = +\infty.$$

So, $\boxed{\int_{-1}^{1} x^{-4}\,dx = +\infty}$ ← We say that the integral "diverges"

More generally,

$$\int_{0}^{1} x^{-p}\,dx = \begin{cases} \frac{1}{1-p} & \text{for } p<1 \text{ — converges} \\ +\infty & \text{for } p\geq 1 \text{ — diverges} \end{cases}$$

THE "p"-TEST



$$\int_{1}^{\infty} x^{-p}\,dx = \begin{cases} +\infty & \text{for } p<1 \text{ — diverges} \\ \frac{1}{p-1} & \text{for } p\geq 1 \text{ — converges} \end{cases}$$

We might need to use both p-tests simultaneously.

Q2. What is $\int_0^\infty 1/x^2\, dx$?

Ans. $\int_0^\infty x^{-2}dx = \underbrace{\int_0^1 x^{-2}dx}_{+\infty} + \underbrace{\int_1^\infty x^{-2}dx}_{\frac{1}{2-1}=1}$

So, the answer is $\underline{\infty}$ (Divergent)

Q3. And $I = \int_0^4 \frac{dx}{\sqrt{4-x}}$?

Ans. Careful! The "trouble" is at 4, not 0. Substitute $u = 4-x$, so $du = -dx$ and

$$I = \int_4^0 \frac{-du}{\sqrt{u}} = -\int_0^4 -u^{-1/2}\,du$$

$$= \int_0^4 u^{-1/2}\,du = \int_0^1 u^{-1/2}\,du + \int_1^4 u^{-1/2}\,du$$

Split into two to use the p-Test!

By p-test, first integral converges and equals $\frac{1}{1-1/2} = 2$. As for the second, there is no "trouble":

$$\int_1^4 u^{-1/2}\,du = \left.\frac{u^{-1/2+1}}{-1/2+1}\right|_{u=1}^{u=4}$$

$$= \left. 2u^{1/2}\right|_{u=1}^{u=4}$$

$$= 2\sqrt{4} - 2\sqrt{1} = 2$$

So, $I = 2+2 = \boxed{4}$

We can use these p-integrals to determine the convergence in some cases even when we can't actually solve the integral... Here's an example Problem. (From Penn Calc Wiki)

Q4 $I_1 = \int_2^{\infty} \frac{dx}{\sqrt{x^3-8}}$ and $I_2 = \int_2^{\infty} \frac{dx}{(x-2)^3}$ Which one converges?

Ans. I_2 is easier! Set $u = (x-2)$, get $du = dx$ so

$$I_2 = \int_0^{\infty} u^{-3}\,du = \int_0^1 u^{-3}\,du + \int_1^{\infty} u^{-3}\,du$$

The first integral already diverges by p-test!

I_1 takes work: We get

$$I_1 = \int_2^{\infty} \frac{dx}{\sqrt{x^3-8}}, \quad \text{so set } u = x^3 - 8.$$

and $x = (u+8)^{1/3}$.

Then, $du = 3x^2 dx$

so, $du = 3(u+8)^{2/3} dx$

Now, $I_1 = \int_0^{\infty} u^{-1/2} \cdot \frac{du}{3(u+8)^{2/3}}$

$$= \frac{1}{3} \int_0^{\infty} u^{-1/2} (u+8)^{-2/3}\,du$$

Want to use two p-tests $\leftarrow$ $= \frac{1}{3}\left[\int_0^1 u^{-1/2}(u+8)^{-2/3}\,du + \int_1^{\infty} u^{-1/2}(u+8)^{-2/3}\,du \right]$

For $u \to 0^+$, TAYLOR: $(u+8)^{-2/3} = 8^{-2/3}(u/8+1)^{-2/3}$

(Binomial!) $= 8^{-2/3} \sum_{k=0}^{\infty} \binom{-2/3}{k} u^k = 8^{-2/3}\left[1 - \frac{2}{3}u + O(u^2)\right]$

So, the integral $\frac{1}{3}\int_0^1 u^{-1/2}(u+8)^{-2/3}\,du$ looks like

$1/3 \int_0^1 u^{-1/2}\,[1+O(u)]\,du$, which

converges by the p-test, since $1/2 < 1$.

For $u \to +\infty$, examine $\frac{1}{3}\int_1^\infty u^{-1/2}(u+8)^{-2/3}\,du =$

$1/3 \int_1^\infty \frac{du}{u^{1/2}(u+8)^{2/3}}$. Now, readjust the stuff

in the denominator: $(u+8)^{2/3} = u^{2/3}\cdot(1+8/u)^{2/3}$

$= u^{2/3}\,(1+O(1/u))$ for large u. So,

our integral is $1/3 \int_1^\infty \frac{du}{u^{1/2}\cdot u^{2/3}(1+O(1/u))}$

$\frac{1}{2}+\frac{2}{3}$

$= 1/3 \int_1^\infty u^{-7/6}(1+O(u^{-1}))\,du$

Again, p-test! $7/6 > 1$, so we get convergence.

So, I_1 converges while I_2 diverges.

TRIG INTEGRALS:

A $\int \sin^m x \cos^n x\,dx$:

if m is odd, $u = \cos x$, and $(1-\sin^2=\cos^2)$

Q5 $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \cdot \sin x\,dx$

Set $u = \cos x$, $du = -\sin x\,dx$

And, $\sin^2 x = 1-\cos^2 x = 1-u^2$

$\int \sin^3 x \cos^2 x\,dx = -\int (1-u^2)u^2\,du$, easy!

Similarly, if n is odd, then set $u = \sin x$, etc.

If both m and n are even, then things get much harder, and we won't worry about that yet.

Q6 $\int \sec^6 x \tan^3 x \, dx = ?$

Ans Pull out a "$\sec^2 x$"...

$$\int \sec^6 x \tan^3 x \, dx = \int \sec^4 x \tan^3 x \underline{\sec^2 x \, dx}$$

Then, set $u = \tan x$, $du = \sec^2 x \, dx$,

and $\sec^4 x = (\sec^2 x)^2 = (1 + \tan^2 x)^2$, so

$$\int \sec^6 x \tan^3 x \, dx = \int (1+u^2)^2 \cdot u^3 \cdot du, \text{ easy!}$$