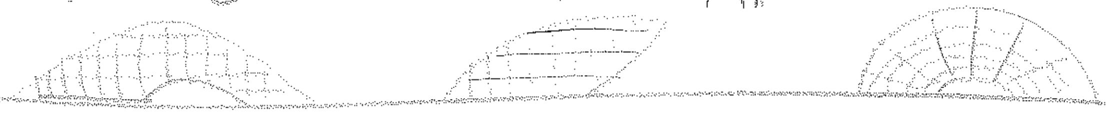
DAY 18
FRI OCT 16

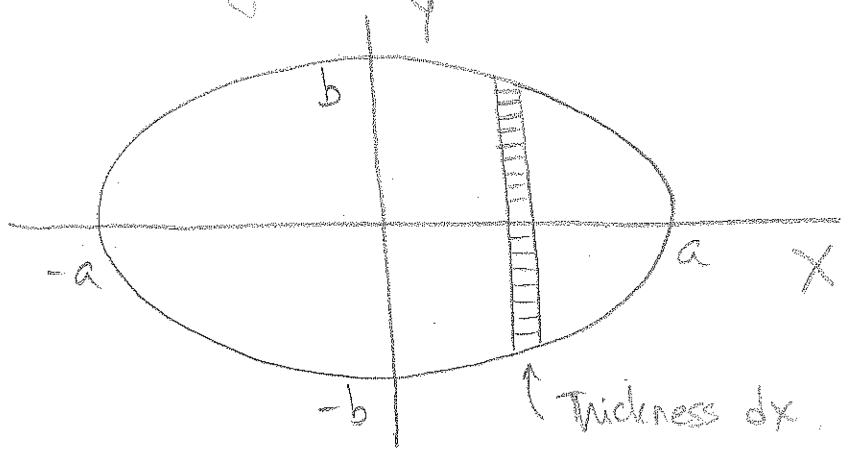
SIMPLE & COMPLEX AREAS



Q1 What is the area of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \ ?$$

Ans PICTURE! Always know the picture!



Slice it vertically: what are the upper and lower endpoints? Solve for y in terms of x:

$$y^2/b^2 = 1 - x^2/a^2$$

$$\text{So} \quad y = \pm b\sqrt{1 - x^2/a^2}$$

So, the strip of thickness "dx" runs vertically from $-b\sqrt{1-x^2/a^2}$ to $+b\sqrt{1-x^2/a^2}$. So,

$$dA = \left[b\sqrt{1-x^2/a^2} - \left(-b\sqrt{1-x^2/a^2}\right)\right] dx$$

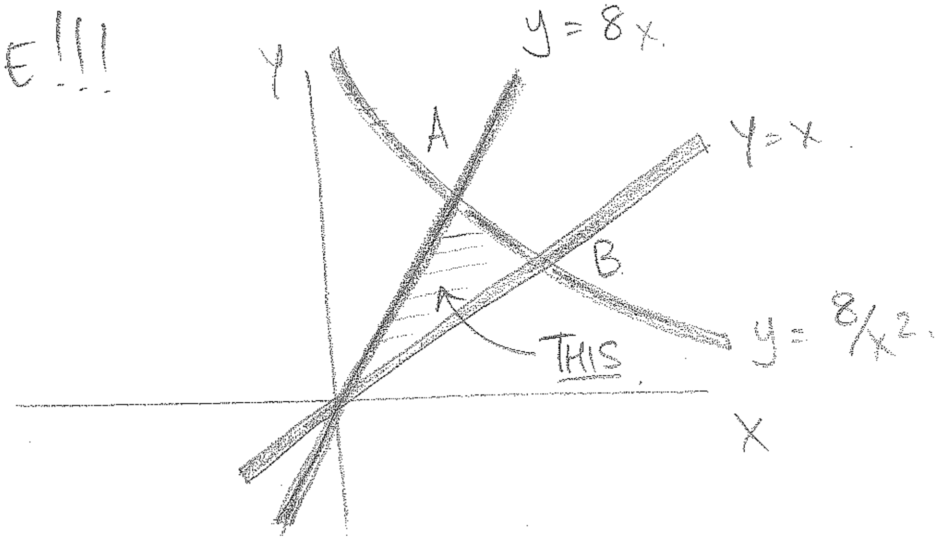
$$= 2b\sqrt{1-x^2/a^2}\, dx$$

So, $A = \int dA = \int_{-1}^{1} 2b\sqrt{1-x^2/a^2}\, dx$

$= 2b\int_{-1}^{1}\sqrt{1-x^2/a^2}\, dx$. Now sub $x = a\sin\theta$ to finish: $A = \pi ab$.

Q2. What is the Area between $y = x$, $y = 8x$ and $y = 8/x^2$?

Ans. PICTURE!!!

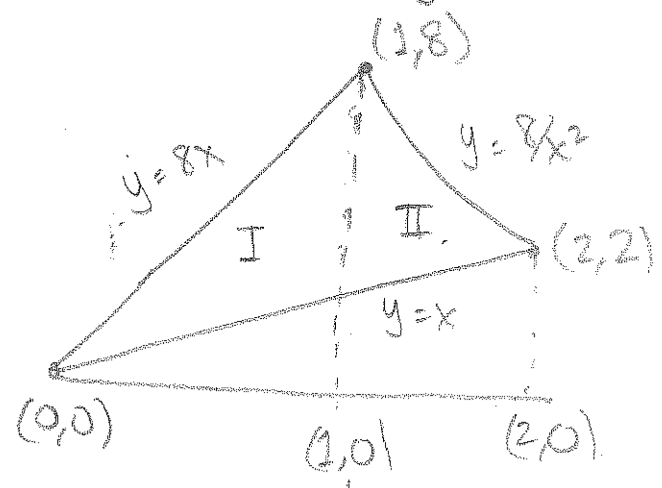


First, need to know A and B:

For A, $8x = 8/x^2$, so $x^3 = 1$, so $\underline{x = 1}$

For B, $x = 8/x^2$, so $x^3 = 8$, so $\underline{x = 2}$

Now, look: the region decomposes into two pieces



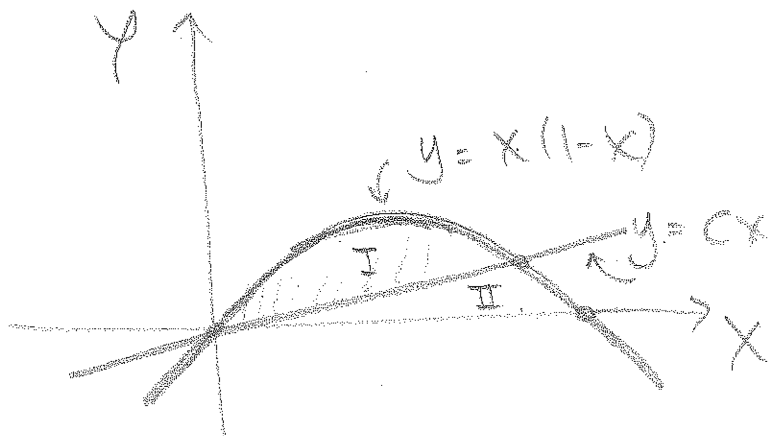
Area of I: $\int_0^1 (8x - x)\,dx = \ldots$

Area of II: $\int_1^2 (8/x^2 - x)\,dx = \ldots$

Solve both, then add 'em up!

Q3 At which value of "c" does the line $y = cx$ cut the area between the x-axis and the curve $y = x(1-x)$ into EXACTLY two halves?

Ans



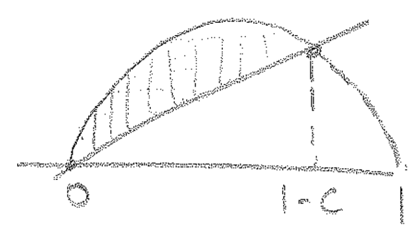
For which c is I = II in Area?

$y = x(1-x)$ intersects $y = cx$ at two points:

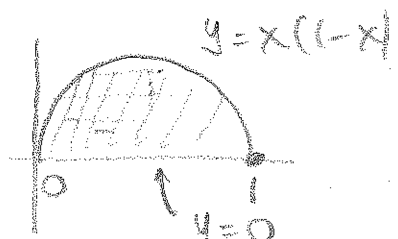
$$x(1-x) = cx$$

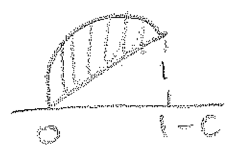
So $x^2 = (1-c)x$

So $x = 0$ or $x = 1-c$.

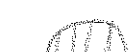


NOTE: when $c = 0$, we get the whole area,



So area of  is:

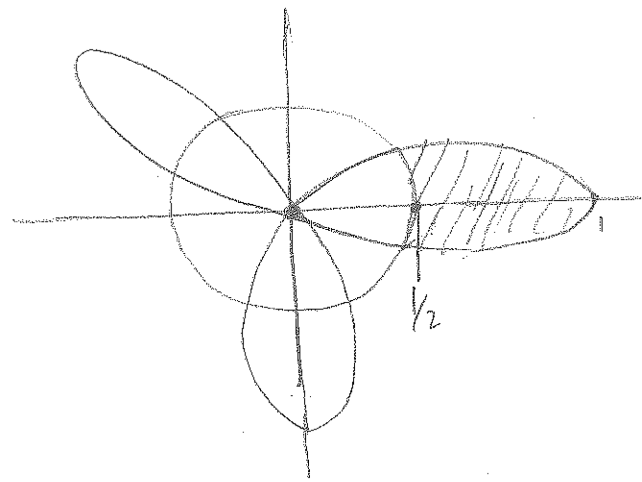
$$\int_0^{1-c} [x(1-x) - cx]\,dx = \int_0^{1-c} [(1-c)x - x^2]\,dx = \left[\frac{(1-c)x^2}{2} - \frac{x^3}{3}\right]_0^{1-c}$$

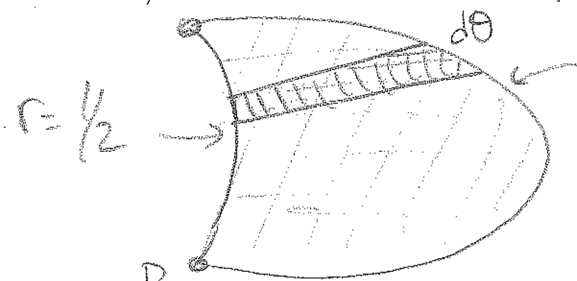
$= \boxed{(1-c)^3 \cdot 1/6}$ So, Area of  $= 1/6$

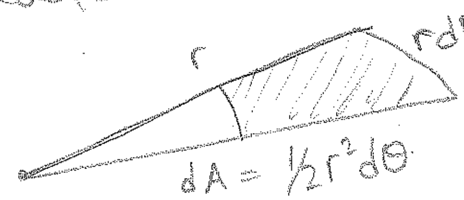
Therefore, want $(1-c)^3 = 1/2$ OR $\boxed{c = 1 - \frac{1}{\sqrt[3]{2}}}$

Q4 [Polar coordinates]. Find the Area OUTSIDE the circle $r = 1/2$ but inside a petal of the curve $r = \cos(3\theta)$

Ans.







Intersection points: $\cos(3\theta) = 1/2$,

so $3\theta = \pm \pi/3$

or $\theta = \pm \pi/9$

$$dA = \frac{1}{2}\left[\cos^2(3\theta) - \frac{1}{4}\right] d\theta$$

So,
$$A = \frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\cos^2(3\theta) - \frac{1}{4}\right) d\theta$$

Now integrate, using $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$.