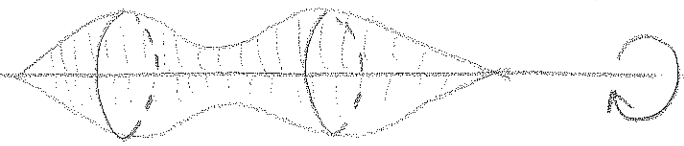
DAY 19
MON OCT 19

VOLUMES OF REVOLUTION

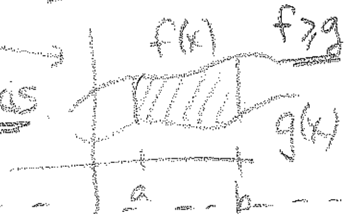


The "ONLY" formula:

$$\boxed{V = \int_{\circ}^{\circ} dV}$$

← Useless, unless you can determine dV and limits

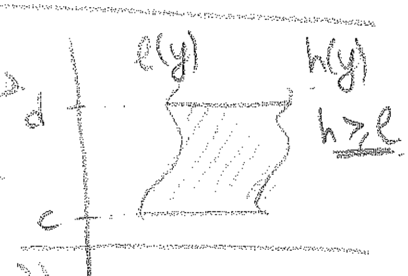
I: Rotate about X-axis



Get cross-section of Area $= \pi(f^2(x) - g^2(x))$

So, $\boxed{V = \pi \int_a^b [f^2(x) - g^2(x)]\,dx}$

II. Rotate about Y-AXIS.

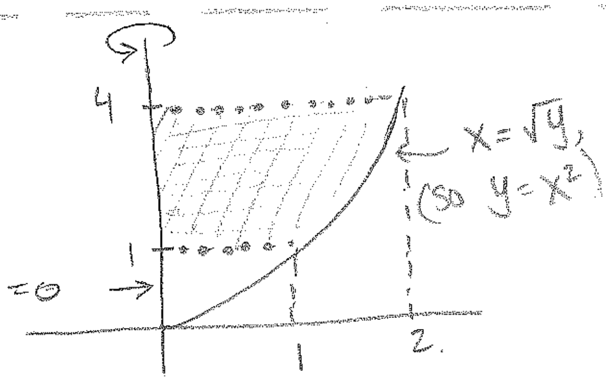


Again, cross-section of area $= \pi(h^2(y) - \ell^2(y))$,

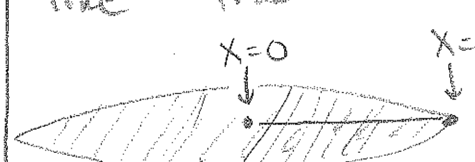
So, $\boxed{V = \pi \int_c^d [h^2(y) - \ell^2(y)]\,dy}$

Q1. Find the volume of the solid obtained by rotating the region between $x = \sqrt{y}$ and $x = 0$ for y in $[1, 4]$ about the Y-axis.

Ans. PICTURE:



Cross-section at y looks like this:



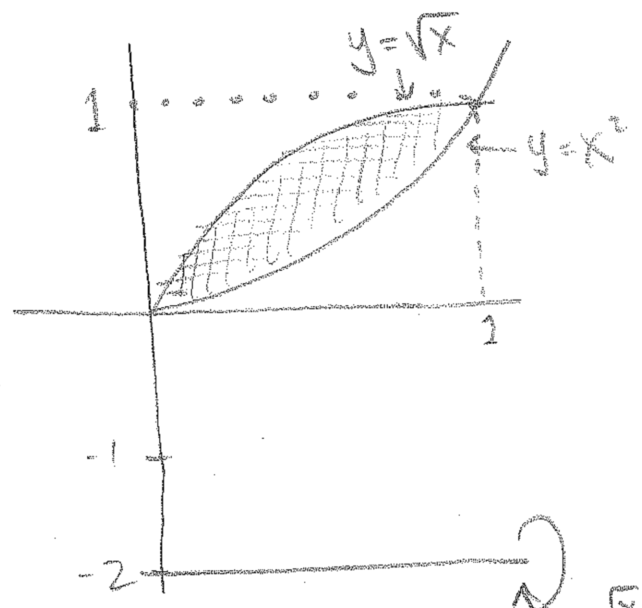
= A disk of radius $\sqrt{y}$, which has area $= \pi y$.

So,
$$V = \int_1^4 \pi y\,dy = \pi \frac{y^2}{2}\Big|_1^4 = \pi\left(\frac{16}{2} - \frac{1}{2}\right)$$
$$= \boxed{\frac{7\pi}{2}}$$

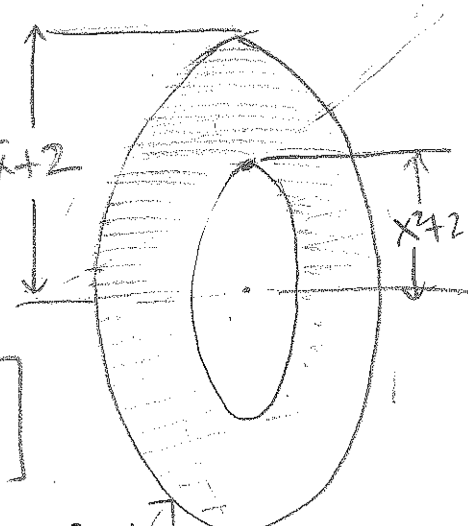
Q2. Region between $y = x^2$ and $y = \sqrt{x}$, rotate about the line $\underline{y = -2}$!

Ans

PICTURE. $\rightarrow$



Now, cross-section (at x) looks like:



and has area

$$A(x) = \pi\left[(\sqrt{x}+2)^2 - (x^2+2)^2\right]$$
$$= \pi\left[x + 4\sqrt{x} + \cancel{4} - x^4 - 2x^2 - \cancel{4}\right]$$
$$= \pi\left[x + 4\sqrt{x} - x^4 - 2x^2\right]$$

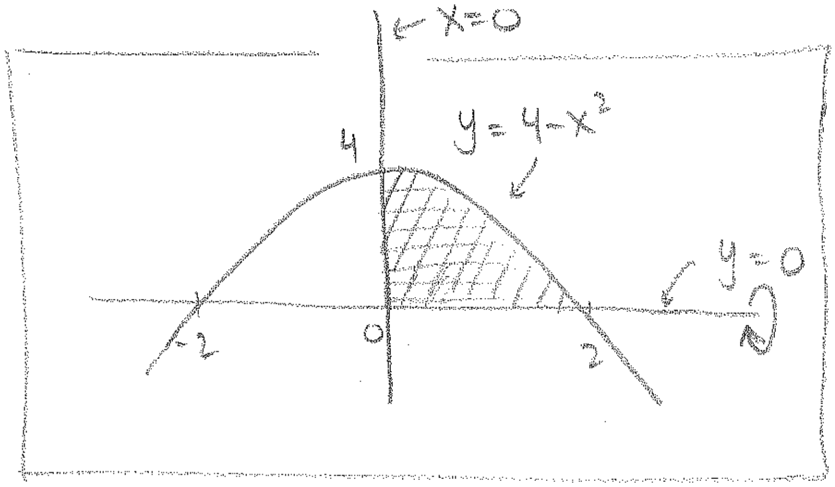
So, $dV = A(x)\,dx$ and finally,

$$V = \pi\int_0^1 \left[x + 4\sqrt{x} - x^4 - 2x^2\right] dx$$
$$= \text{(easy integral!)}$$

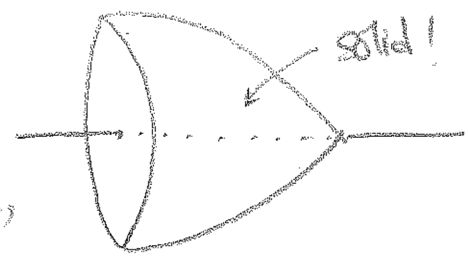
In previous examples, the CROSS-SECTIONS were chosen in the direction PERPENDICULAR to the axis of rotation, which gave us (differences of) disks. If we choose PARALLEL cross sections we get cylinders instead...

Q3. Compute the volume of the solid given by rotating the region between $y = 4 - x^2$, $x = 0$ and $y = 0$ about the x-axis

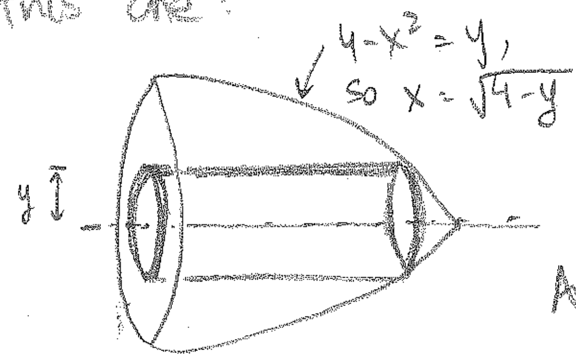
Ans PICT... you get the idea:



We get something like



, and want to decompose it into concentric cylinders, like this one:



Radius $= y$

Height $= \sqrt{4-y}$

$$\text{Area} = 2\pi R H = 2\pi y \sqrt{4-y}$$

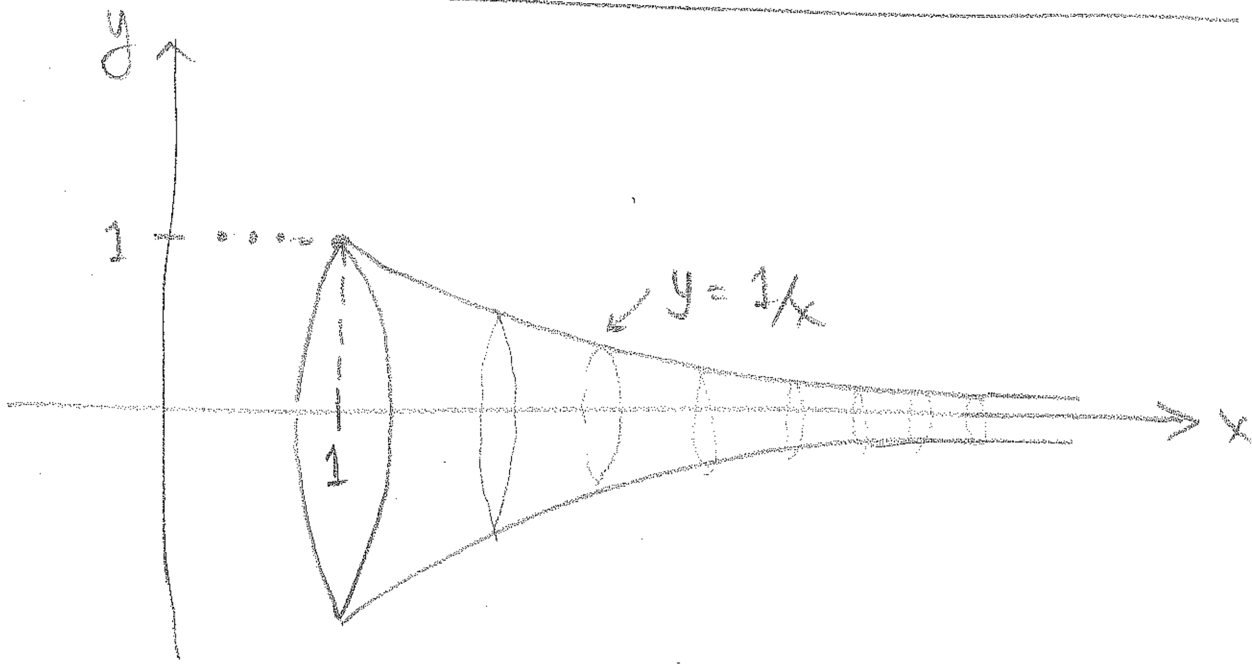
$$dV = A(y)\,dy = 2\pi y \sqrt{4-y}\,dy$$

So,
$$V = 2\pi \int_0^4 y\sqrt{4-y}\,dy$$

Use substitution $u = 4 - y$, easy!

Q4 | [The HORN OF GABRIEL] What are the surface area and volume of the region generated by rotating $y = 1/x$ about the x-axis between 1 and ∞?

Ans



First, VOLUME: the cross-section over x is a disk of radius $1/x$, with area π/x^2.

So, $dV = \pi x^{-2}\,dx$, and

$$V = \int_1^\infty \pi x^{-2}\,dx$$

$$= \pi \int_1^\infty x^{-2}\,dx = \pi\cdot(1)$$

Type-B p-integral $p=2$!!

So, volume $= \underline{\pi}$.

Next, AREA: the cross-section is a CIRCLE of radius $1/x$, so $dA = 2\pi x^{-1}\,dx$

$$A = \int_1^\infty 2\pi x^{-1}\,dx = 2\pi \int_1^\infty x^{-1}\,dx$$

But $\int_1^\infty x^{-1}\,dx = \infty$, p-test again.

So, GABRIEL'S HORN has FINITE VOLUME But INFINITE AREA!