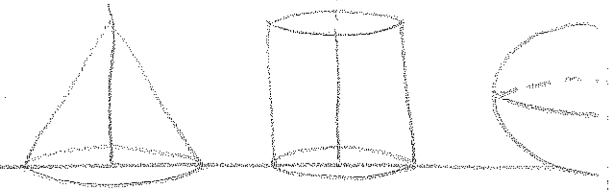
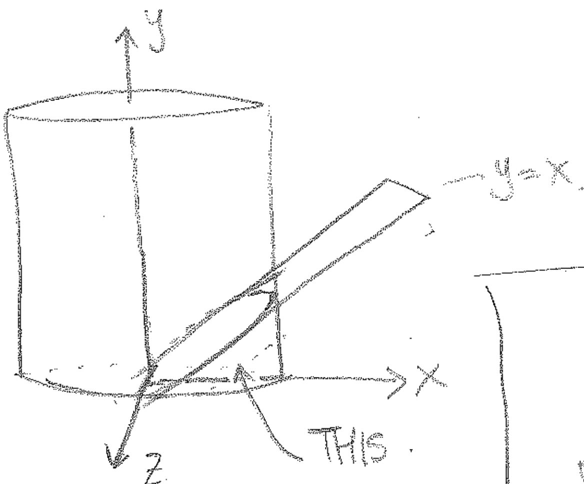
DAY 20
WED OCT 21

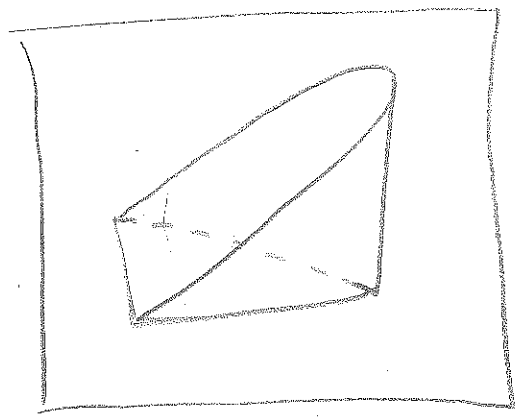
More Volumes!



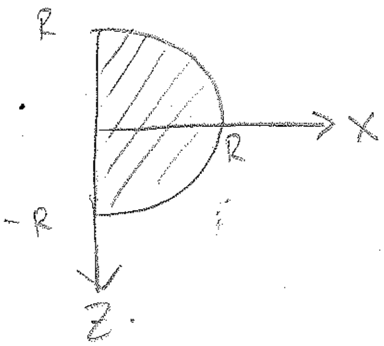
Q1 Find the volume of the region contained in the half cylinder $\{x^2+y^2 \le R^2,\ y \ge 0\}$ above the x-axis and below the plane $y = x$.

Ans.

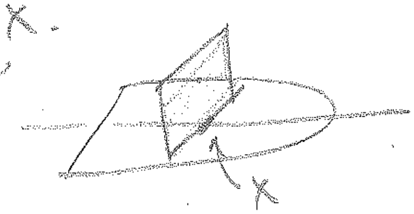




The base is a half-disk of radius R.



And on top of each vertical line, the cross-section is a RECTANGLE: of $\underbrace{\text{width } \sqrt{R^2-x^2}}_{\text{from the disk.}}$ and $\underbrace{\text{height } x.}_{\text{(from the equation } y=x\text{)}}$

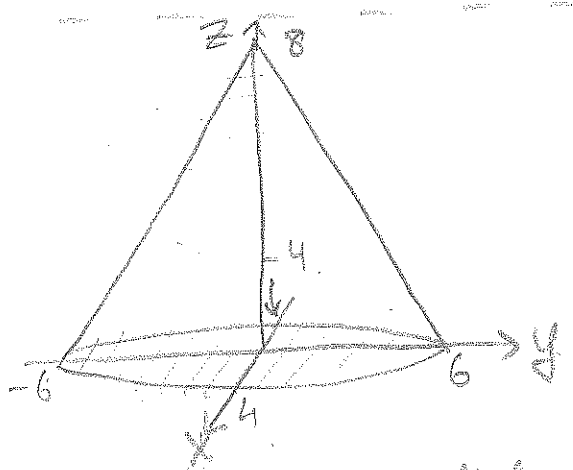


So, $dV = x\sqrt{R^2-x^2}\,dx$, for x from 0 to R

Therefore, $V = \int_0^R x\sqrt{R^2-x^2}\,dx$, (sub $u = x^2$)

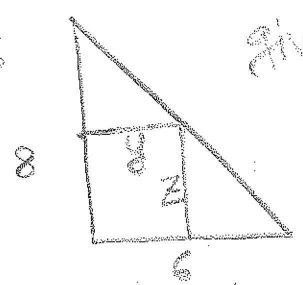
Q2 Find the volume of the cone of height 8 based on the ellipse with major radius = 6 and minor radius = 4.

Ans.



Let's chop into ellipses parallel to the xy plane: at height $z = 0$, major axis = 6, minor axis = 4.

But at height $z \leq 8$, in general the major axis is given by



$\frac{8-z}{8} = y/6$

so, $y = 6 - \frac{3z}{4}$

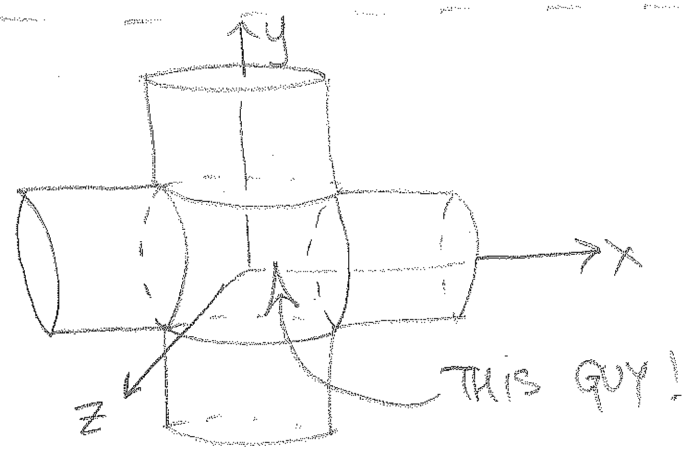
and the minor axis by $\frac{8-z}{8} = y/4$, so $y = 4 - z/2$. So, AREA OF ELLIPSE at height z is $\pi(6-3z/4)(4-z/2) = \pi\left[\frac{3z^2}{8} - 6z + 24\right]$

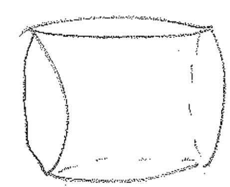
So, $dV = \pi\left[\frac{3z^2}{8} - 6z + 24\right]$, ($z$ from 0 to 8)

Thus, $V = \pi\int_0^8 \left(\frac{3z^2}{8} - 6z + 24\right) dz$ = (easy!)

Q3 [WIKI MONSTER] Volume of intersection of $x^2+y^2 \le R^2$ AND $y^2+z^2 \le R^2$.

Ans



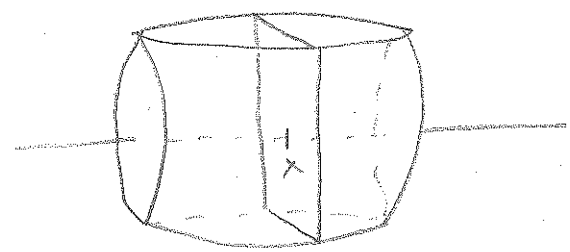


The cylinders are $x^2+y^2 = R^2$ and $y^2+z^2 = r^2$, so at their intersection we get:

x : $-R$ to R.

z : $-\sqrt{R^2-x^2}$ to $\sqrt{R^2-x^2}$.

Cross-sections over x are SQUARES, i.e. of area $(2z)^2 = 4z^2 = 4(R^2-x^2)$

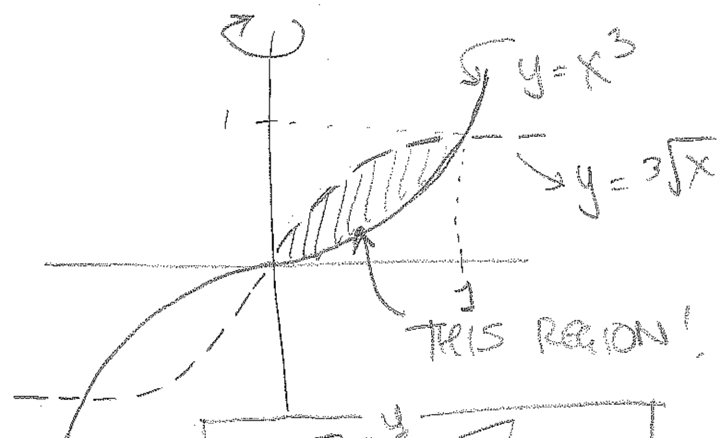


So, $dV = 4(R^2-x^2)\,dx$, for x from $-R$ to R.

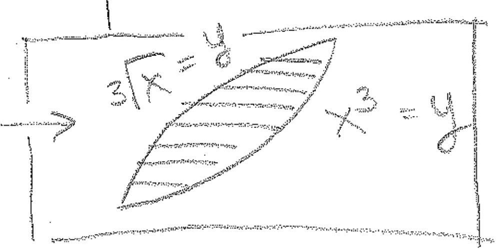
$\Rightarrow V = 4\int_{-R}^{R} (R^2-x^2)\,dx$, easy!

Q4 Vol of region between $y = x^3$, $y = \sqrt[3]{x}$ and $x \geq 0$ rotated about y-axis

Ans



Slice horizontally →



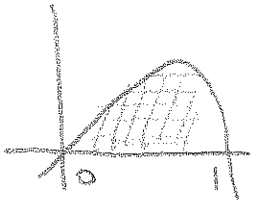
At height y, we get annular cross-sections of inner radius y^3 and outer radius $y^{1/3}$. So,

$$dV = \pi \left(y^{2/3} - y^6\right) dy,$$

and $$V = \pi \int_0^1 \left(y^{2/3} - y^6\right) dy,$$ easy!

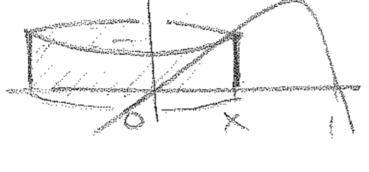
Q5 [WHY WE NEED CYLINDRICAL SHELLS]: Find the volume of the region obtained by rotating the area "under $y = x - x^5$ and above the x-axis" all about the y-axis.

Ans



We can't chop horizontally, since finding the cross-sections requires solving $y = x - x^5$ for x! But, we chop VERTICALLY.

Get cylinder of radius $= x$, height $= x - x^5$, so area $= 2\pi x(x - x^5)$. Therefore,



$$dV = 2\pi (x^2 - x^6) dx,$$ and now

$$V = 2\pi \int_0^1 (x^2 - x^6) dx,$$ (easy!)