DAY 21
FRI OCT 23

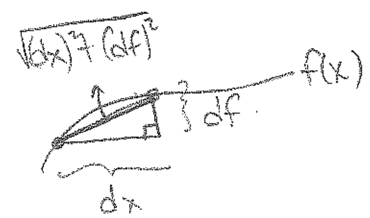
ARCLENGTH, SURFACE AREA

BASIC IDEA

The "length element" $d\ell$ on the graph of $y = f(x)$ is given by

$$\boxed{d\ell = \sqrt{1+[f'(x)]^2}}$$

CAREFUL... This is the SQUARE of the first derivative, NOT the second derivative!



Q1. Calculate the arclength of $y = x^3/12 + 1/x$ for $1 \le x \le 2$

Ans

$$y = x^3/12 + 1/x \,, \quad \text{so} \quad dy/dx = x^2/4 - 1/x^2$$

So,
$$\left(dy/dx\right)^2 = \left(x^2/4 - 1/x^2\right)^2 = x^4/16 - 2\left(x^2/4\right)\left(1/x^2\right) + 1/x^4$$
$$= x^4/16 - 1/2 + 1/x^4$$

and,
$$d\ell = \sqrt{1+(dy/dx)^2} = \sqrt{1+\left(x^4/16 - 1/2 + 1/x^4\right)}$$
$$= \sqrt{x^4/16 + 1/2 + 1/x^4}$$

Now,
$$\ell = \int_1^2 \sqrt{x^4/16 + 1/2 + 1/x^4}\, dx$$

The ONLY WAY you'll be able to solve this (fast) is if you realize that $x^4/16 + 1/2 + 1/x^4 = \left(x^2/4 + 1/x^2\right)^2$, and so $\ell = \int_1^2 \left(x^2/4 + 1/x^2\right) dx$, EASY!

Q2. Can you compute the arclength of $y = x^4 - x$ between -1 and 1?

Ans No, but I can set up the integral which would compute it!

$$dy/dx = 4x^3 - 1, \quad \text{so}$$

$$d\ell = \sqrt{1 + (4x^3-1)^2}$$

$$= \sqrt{1 + 16x^6 - 8x^3 + 1}$$

$$= \sqrt{16x^6 - 8x^3 + 2}.$$

So, $$\ell = \int_{-1}^{1} \sqrt{16x^6 - 8x^3 + 2}\, dx.$$ ← Good luck solving that...

SOMETIMES, the curve is not the graph of a function, but rather a parametrized one, eg:

CIRCLE : $c(t) = (r\cos t, r\sin t)$ for $0 \le t \le 2\pi$.

ELLIPSE : $e(t) = (a\cos t, b\sin t) = (x(t), y(t))$.

Here, $d\ell = [(x'(t))^2 + (y'(t))^2]^{1/2}$.

Q3. What is the arclength of $(a\cos t, b\sin t)$ for t between 0 and π?

Ans $x(t) = a\cos(t)$, $\quad y(t) = b\sin(t)$

so $x'(t) = -a\sin(t)$ $\quad y'(t) = b\cos t$

$$d\ell = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}.$$

And, $$\ell = \int_0^{\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$ (UGH!!)

We can only solve this in VERY EASY cases, eg
$a=0, \quad b=0, \quad a=b$.

Q4. Arclength of a function $F(x)$ whose DERIVATIVE is $dF/dx = \sqrt{e^x(e^x+2)}$, for $0 \le x \le 1$.

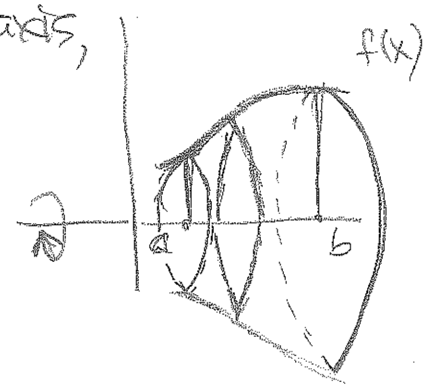
Ans

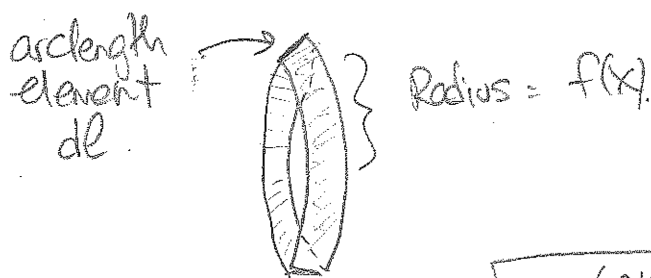
$$d\ell = \sqrt{1+(dF/dx)^2} = \sqrt{1+e^x(e^x+2)}$$
$$= \sqrt{1+e^{2x}+2e^x} = \sqrt{(1+e^x)^2}$$
$$= 1+e^x \qquad \text{(Aha!!)}$$

So, $$\ell = \int_0^1 (1+e^x)\,dx = x+e^x \Big|_{x=0}^{x=1}$$
$$= (1+e) - (0+1) = \boxed{e}$$

SURFACE AREA:

When rotating $y=f(x)$ about x-axis, the area element dA is this:





So, $dA = 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ and so

$$\boxed{A = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx}$$

(All the problems with arclength integrals, ...)

Similarly, if $x = g(y)$ and we rotate about y-axis from $c \le y \le d$, then

$$\boxed{A = 2\pi \int_c^d g(y) \sqrt{1 + (g'(y))^2}\, dy}$$

Q5 | What is the surface area obtained by rotating $y = \sin(x)$ about the x-axis for $0 \le x \le \pi$?

Ans | $dy/dx = \cos(x)$, so length element $d\ell = \sqrt{1+\cos^2(x)}\, dx$. Now,

$$dA = 2\pi y\, d\ell$$
$$= 2\pi \sin(x) \sqrt{1+\cos^2(x)}\, dx, \quad \text{so}$$

$$A = 2\pi \int_0^{\pi} \sin(x) \sqrt{1+\cos^2(x)}\, dx.$$

Set $u = \cos(x)$, $du = -\sin(x)\, dx$ and so

$$A = -2\pi \int_1^{-1} \sqrt{1+u^2}\, du.$$

Can you solve this? (Hint, sub $u = \sinh\theta$).

Q6 | Area: rotating e^{-x} about x-axis for $0 \le x \le 1$.

Ans | $d\ell = \sqrt{1 + (-e^{-x})^2}\, dx = \sqrt{1 + e^{-2x}}\, dx$

$$dA = 2\pi e^{-x} \sqrt{1+e^{-2x}}\, dx$$

So, $A = 2\pi \int_0^1 e^{-x} \sqrt{1+e^{-2x}}\, dx$

Set $u = e^{-x}$, so $\boxed{A = -2\pi \int_1^{1/e} \sqrt{1+u^2}\, du}$