DAY 22
MON OCT 25

# WORK, ELEMENTS

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BASIC IDEA: To compute a locally-varying quantity $Q$, find $dQ$ and integrate:

$$\boxed{Q = \int dQ}$$

When $Q$ is work, then $dQ = dW = \boxed{F(x)\,dx}$, where $F(x)$ is "force at $x$" and $dx$ is displacement.

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**Q1** If a linear spring has constant $10\ \mathrm{N/m}$ and is initially stretched to $1\mathrm{m}$, how much further can it be stretched by $20\ \mathrm{Nm}$ of work?

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**Ans** For spring constant $K$, $dW = F(x)\,dx = Kx\,dx$.
So, to stretch from $a$ to $b$ requires

$$W = \int_a^b Kx\,dx = \frac{K}{2}(b^2-a^2) \text{ Nm of work}$$

For us, "$b$" is unknown. But:

$$W = 20\ \mathrm{Nm}, \quad a = 1\mathrm{m}, \quad K = 10\ \mathrm{N/m}$$

So, $$20 = \tfrac{10}{2}(b^2-1)$$

So, $b^2 - 1 = 4$, so $\boxed{b = \sqrt{5}\ \mathrm{m}}$

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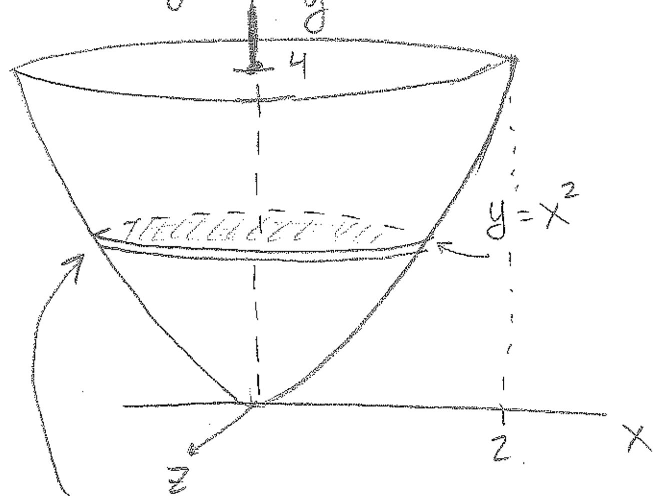
**Q2** Let $R$ be the region obtained by rotating the graph of $y = x^2$ around the $y$-axis for $0 \le x \le 4\mathrm{m}$. How much work does it take to dig a ditch of this shape if we assume the dirt has constant density?

Ans

Here's the region:




At height $y$ above the origin, the cross-sectional disk has radius $x = \sqrt{y}$, hence area $A(y) = \pi(\sqrt{y})^2 = \pi y$. So,

$$dV = \pi y \, dy.$$ (volume element)

Using constant density $\rho$ for the dirt and a constant "$g$" acceleration due to gravity, we get:

$$dM = \rho dV = \rho . \pi y \, dy$$ (Mass element = density x Volume.)

$$dF = (dM) g = \rho g \pi y \, dy$$ (force element = Mass x acceleration)

Finally, displacement of this cross-section is $(4-y)$: it has to be taken to the Top. So,

$$dW = \rho g \pi y (4-y) \, dy,$$ ($y$ from 0 to 4)

Thus, $$W = \rho g \pi \int_0^4 y(4-y) \, dy = \rho g \pi \int_0^4 (4y - y^2) \, dy$$

$$= \rho g \pi \left[ 2y^2 - y^3/3 \right]_0^4$$

$$= \rho g \pi \left[ 32 - 64/3 \right]$$

$$= \boxed{32/3 \, \rho g \pi} \text{ Nm.}$$

**Q3** To what height $H$ must we dig an upside-down CONE of radius 2 containing the same dirt as Q2 so that we do only half as much work?

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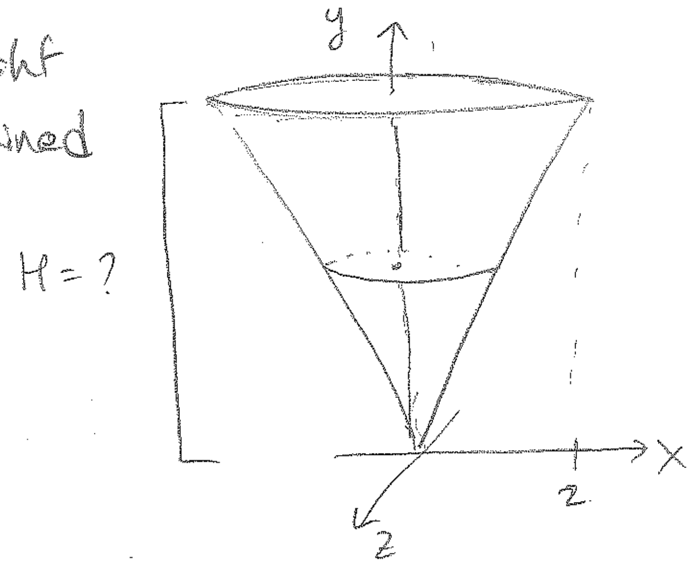
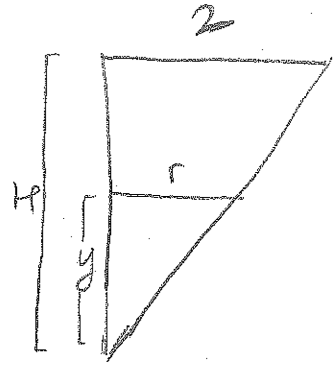
**Ans** The cross-section at height $y$ has radius $r$ determined by the following similar triangles:




$$y/r = H/2, \quad \text{so} \quad r = 2y/H.$$

Thus, $A(y) = \pi r^2 = \frac{4\pi}{H^2} y^2$, and

$$dV = \frac{4\pi}{H^2} y^2\, dy, \quad \text{so} \quad dM = \frac{4\pi \rho}{H^2} y^2\, dy.$$

Finally, displacement $= (H-y)$, so we get

$$dW = \frac{4\pi\rho g}{H^2} y^2 (H-y), \quad \text{and so}$$

$$W = \int_0^H \frac{4\pi\rho g}{H^2} y^2 (H-y)\, dy$$

$$= \frac{4\pi\rho g}{H^2} \int_0^H (Hy^2 - y^3)\, dy$$

$$= \frac{4\pi\rho g}{H^2} \left[ \frac{Hy^3}{3} - \frac{y^4}{4} \right]_0^H$$

$$= 4\pi\rho g \cdot H^2 \left[ \frac{1}{3} - \frac{1}{4} \right] = \boxed{\frac{\pi\rho g}{3} H^2} \ \text{Nm}$$

Comparing to Q2, we need to find $H$ so that

$$\cancel{\frac{\pi \rho g}{3}} H^2 = \frac{1}{2} \pi \rho g \cdot \frac{32}{\cancel{3}}$$

So, $H^2 = 16$, or $\boxed{H = 4}$

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## OTHER ELEMENTS :

**Mass:** $dM = \rho \, dV$ (ρ: density, $V$: volume)

**Torque:** $dT = x \, dF$ ($x$: distance, $F$: force)

**Force:**

$$dF = dM \cdot a$$

(M: mass, a: acceleration)

$$= P \cdot dA$$

(P: Pressure, A: Area)

**Present Value**

$$dPV = e^{-rt} I(t) \, dt$$

$r$: interest rate

$I(t)$: income stream.

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**Q4** Consider an income stream that pays quadratically $I(t) = t^2$ for all time $t$. What is the present value if interest rate is 2% throughout?

**Ans.**

$$dPV = I(t) e^{-rt} \, dt = t^2 e^{-rt} \, dt.$$

$$\text{So, } PV = \int_0^\infty t^2 e^{-rt} \, dt$$

Integrate by parts ... or table:

| | | |
|---|---|---|
| $t^2$ | $+$ | $e^{-rt}$ |
| $2t$ | $-$ | $-\frac{1}{r} e^{-rt}$ |
| $2$ | $+$ | $+\frac{1}{r^2} e^{-rt}$ |
| $0$ | | $-\frac{1}{r^3} e^{-rt}$ |

$$PV = \left[ -\frac{t^2}{r} e^{-rt} - \frac{2t}{r^2} e^{-rt} - \frac{2}{r^3} e^{-rt} \right]_{t=0}^{t=\infty}$$

As $t \to \infty$, all terms go to zero, since $e^{-rt}$ will dominate every polynomial (eg: $t, t^2$). As $t \to 0$, only the last term is nonzero. So,

$$PV = 2/r^3 = 2/(0.02)^3 = \boxed{\$\,250{,}000}$$