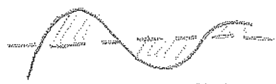
DAY 23
WED OCT 28

AVERAGES AND CENTROIDS



The "AVERAGE" value of $f(x)$ on $[a,b]$ is the NUMBER $\overline{f}$ which satisfies

$$\int_a^b \left[f(x) - \overline{f}\,\right] dx = 0,$$

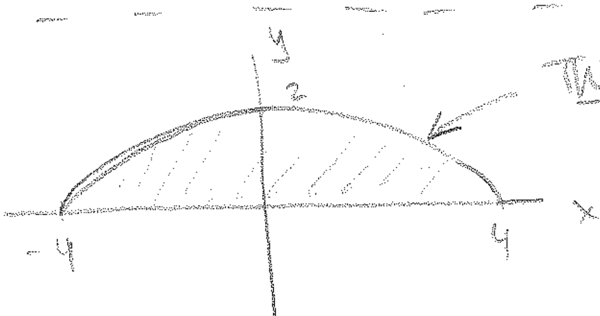
so

$$\boxed{\overline{f} = \frac{\int_a^b f(x)\,dx}{\int_a^b dx}}$$

(so, denominator $= b-a$)

Q1. What is the average value of the function which describes the upper half of the ellipse with x-radius 4 and y-radius 2 ?

Ans



This function: $x^2/4^2 + y^2/2^2 = 1$,

so, $y^2 = 4\left(1 - x^2/16\right)$

so, $y = \sqrt{4 - x^2/4}$

Since $f(x) = \sqrt{4 - x^2/4}$ for $-4 \leq x \leq 4$,

$$\overline{f} = \frac{\int_{-4}^{4} \sqrt{4 - x^2/4}\,dx}{\int_{-4}^{4} dx}$$

$$= 1/8 \int_{-4}^{4} \sqrt{4 - x^2/4}\,dx$$

Now, TRIG SUB $x = 4\cos\theta$ and use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

to get $\boxed{\overline{f} = \pi/2}$

Q2. What is the average value of $f(x) = x\sin x$ for x from 0 to $\pi/2$?

Ans. No need to draw picture, everything is given!

$$\bar{f} = \frac{\int_0^{\pi/2} x\sin x \, dx}{\int_0^{\pi/2} dx}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin x \, dx.$$

Integrate by parts:

$u = x,$ $\quad dv = \sin x \, dx$

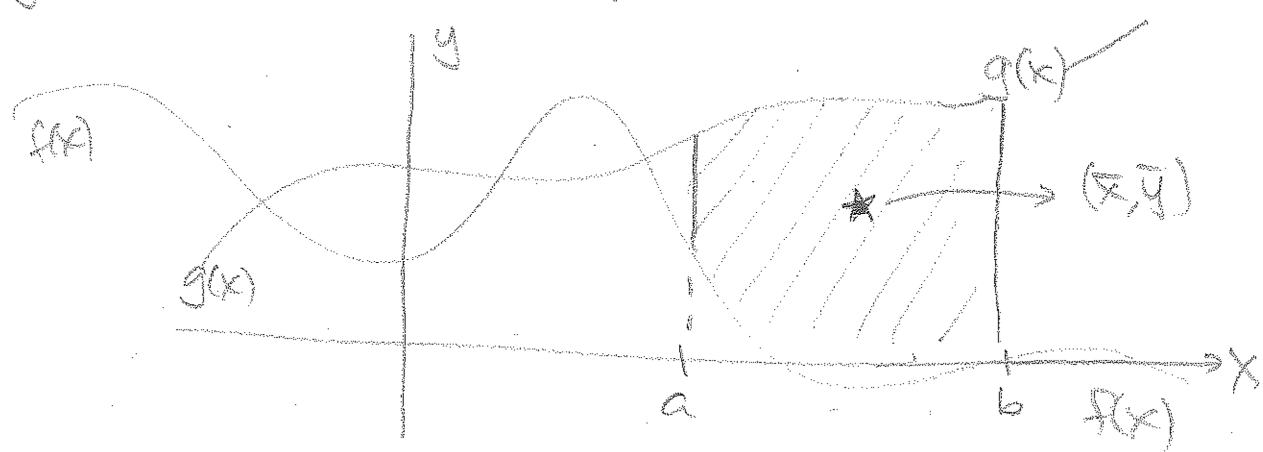
$du = 1,$ $\quad v = -\cos x.$

$$\bar{f} = \frac{2}{\pi}\left[-x\cos x \Big|_{x=0}^{x=\pi/2} + \int_0^{\pi/2} \cos x \, dx \right]$$

$$= \frac{2}{\pi}\left[0 + \sin x \Big|_{x=0}^{x=\pi/2} \right] = \frac{2}{\pi}[1] = \boxed{\frac{2}{\pi}}$$

CENTROID

Given a region between $f(x)$ and $g(x)$ for x from a to b, the "centroid" $(\bar{x}, \bar{y})$ consists of the x-average and the y-average.



Now,

$$\bar{x} = \frac{\int_a^b x\,(g(x) - f(x))\,dx}{\int_a^b (g(x) - f(x))\,dx}$$

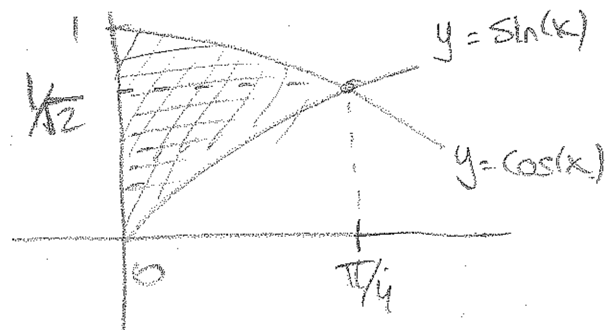
and

$$\bar{y} = \frac{\frac{1}{2}\int_a^b (g^2(x) - f^2(x))\,dx}{\int_a^b (g(x) - f(x))\,dx}$$

[In both cases, denominator = Area of region]

Q3 Find the centroid of the region lying between $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le \pi/4$

Ans



Well; first: Area $= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx$

$= \left(\sin(x) + \cos(x)\right)_{x=0}^{x=\pi/4}$

$= \left(1/\sqrt{2} + 1/\sqrt{2}\right) - (0+1) = \underline{\underline{(\sqrt{2} - 1)}}$

Now, for $\bar{x}$,

$$\bar{x} = \frac{1}{\text{Area}} \int_0^{\pi/4} x(\cos x - \sin x)\,dx$$

$$= \frac{1}{\sqrt{2}-1} \int_0^{\pi/4} x(\cos x - \sin x)\,dx = \frac{\sqrt{2}\,\pi - 4}{4(\sqrt{2}-1)}$$

By parts: set $u = x$, $dv = \cos x - \sin x$, and note that v is already computed!

$\bar{y}$ is easier:

$$\bar{y} = \frac{1}{2(\text{Area})} \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx$$

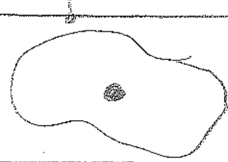
$$= \frac{1}{2(\sqrt{2}-1)} \int_0^{\pi/4} \cos(2x)\, dx$$

$$= \frac{1}{2(\sqrt{2}-1)} \cdot \frac{1}{2} \Big(\sin(2x)\Big)_{x=0}^{x=\pi/4}$$

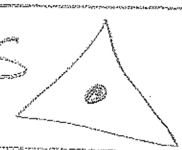
$$= \frac{1}{4(\sqrt{2}-1)} (1-0) = \frac{1}{4(\sqrt{2}-1)}$$

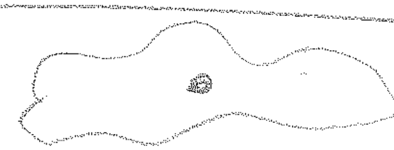
So, center of mass $= (\bar{x}, \bar{y})$

$$= \left(\frac{\sqrt{2}\pi - 4}{4(\sqrt{2}-1)}, \frac{1}{4(\sqrt{2}-1)} \right)$$



CENTER OF MASS





Just like centroids, but with <u>variable density</u> $\rho(x)$. So,

$$\bar{x} = \frac{\int_a^b \underline{\rho(x)} \cdot x\,(g(x) - f(x))\, dx}{\int_a^b \underline{\rho(x)} \cdot (g(x) - f(x))\, dx}$$

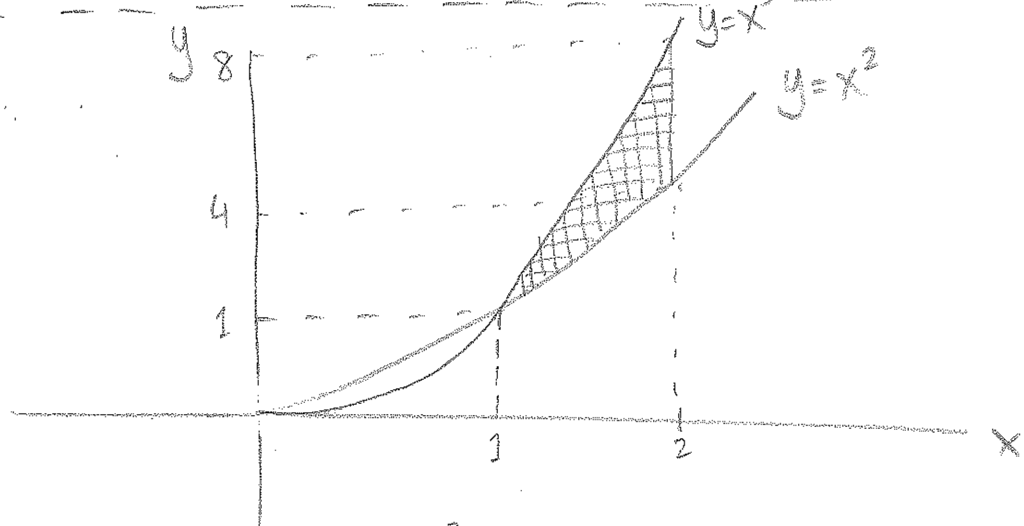
$$\bar{y} = \frac{\frac{1}{2} \int_a^b \underline{\rho(x)} \left(g^2(x) - f^2(x)\right)}{\int_a^b \underline{\rho(x)}\, (g(x) - f(x))\, dx}$$

$\boxed{\text{Denominators are mass, integrals of density}}$

[Same as before, but $\rho(x)$ enters all the integrands!]

Q4. If density $\rho(x)$ is $1/x$, find the center of mass between x^2 and x^3 for $1 \le x \le 2$

Ans.



First, mass $M = \int_1^2 (x^3 - x^2)\, \rho(x)\, dx$

$$= \int_1^2 (x^3 - x^2) \frac{1}{x}\, dx$$

$$= \int_1^2 (x^2 - x)\, dx$$

$$= \left(x^3/3 - x^2/2 \right)_{x=1}^{x=2} = \underline{5/6}.$$

Now, $\bar{x} = \frac{1}{M} \int_1^2 x\, (x^3 - x^2)\, \rho(x)\, dx$

$$= \frac{6}{5} \int_1^2 x\, (x^3 - x^2) \frac{1}{x}\, dx = 6/5 \left[x^4/4 - x^3/3 \right]_{x=1}^{x=2}$$

$$= 6/5 \cdot 17/12 = \underline{17/10}.$$

And, $\bar{y} = \frac{1}{M} \cdot \int_1^2 (x^6 - x^4) \cdot \rho(x)\, dx = \frac{6}{5} \int_1^2 (x^6 - x^4) \frac{dx}{x}$

$$= \frac{6}{5} \int_1^2 (x^5 - x^3)\, dx = \frac{6}{5} \left(x^6/6 - x^4/4 \right)_{x=1}^{x=2}$$

$$= \frac{6}{5} \cdot \frac{25}{4} = \underline{15/2}$$

So, Ans $= \boxed{(17/10,\ 15/2)}$