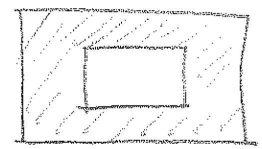
DAY 24
FRI OCT 30

MOMENTS OF INERTIA

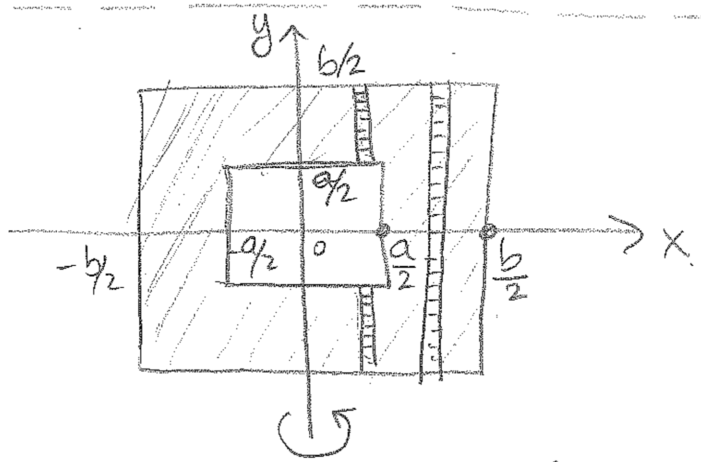
This is the "obstruction to rotational motion", with element $\boxed{dI = dM \cdot r^2}$

dM: Mass element
r: distance to axis of rotation.

Q1. Compute the MoI of this object: a difference of concentric squares with inner side length $= a$, outer side length $= b$, about the vertical axis. Assume constant density ρ:



Ans



At a distance of "x" from the origin, the mass element looks like

$$dM = \begin{cases} 2\dfrac{(b-a)}{\cancel{2}}\rho\, dx, & \text{if } 0 \le |x| \le a \\ \dfrac{2b}{\cancel{2}}\rho\, dx, & \text{if } a \le |x| \le b. \end{cases}$$

Now, $dI = dM \cdot x^2$ and we need three integrals:

$$I = \int_{-a/2}^{a/2} (b-a)\rho x^2 dx + \int_{-b/2}^{-a/2} b\rho x^2 dx + \int_{a/2}^{b/2} 2b\rho x^2 dx$$

(we can split first integral and simplify down to two integrals)

$$I = -\int_{-a/2}^{a/2} a\rho x^2 dx + \int_{-b/2}^{b/2} b\rho x^2 dx$$

$$= -a\rho \cdot x^3/3 \Big|_{x=-a/2}^{x=a/2} + b\rho \cdot x^3/3 \Big|_{x=-b/2}^{x=b/2}$$

$$= \rho/12 \, (b^4 - a^4)$$

Now, mass $M = \rho(b^2 - a^2)$, and

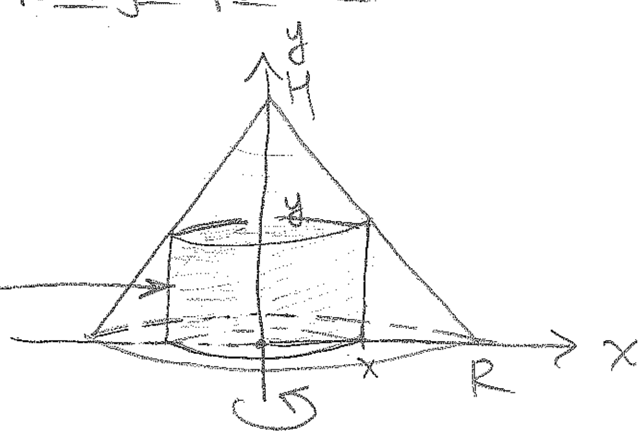
$(b^4 - a^4) = (b^2 - a^2)(b^2 + a^2)$,

so $\boxed{I = \dfrac{M(b^2+a^2)}{12}}$

At HOME, try the <u>same</u> <u>computation</u> but with VARIABLE density $\rho(x) = 2/x$.

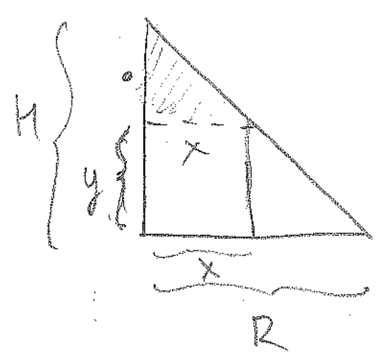
Q2. Find the MoI of a cone with Radius R and height H, rotated about central axis; <u>constant</u> <u>density</u> ρ

Ans.



At distance x from the axis, we have a <u>cylinder</u> of <u>radius x</u> and <u>height y</u>:

(<u>Similar triangles</u>:)



$$\frac{H-y}{x} = \frac{H}{R},$$

so $\underline{y = \frac{H}{R}(R-x)}$

So, our cylinder has volume element

$$dV = 2\pi \underbrace{x}_{\text{radius}} \cdot \underbrace{H/R\,(R-x)}_{\text{height}}\ dx$$

$$= \frac{2\pi H}{R}\left(Rx - x^2\right) dx$$

So, $dM = \rho\, dV = \frac{2\pi\rho H}{R}\left(Rx - x^2\right) dx$

Finally, $dI = dM \cdot x^2$

$$= \frac{2\pi\rho H}{R}\left(Rx^3 - x^4\right) dx$$

And, $I = \int_0^R \frac{2\pi\rho H}{R}\left(Rx^3 - x^4\right) dx$

Not $-R$, we already considered a full cylinder at distance R

$$= \frac{2\pi\rho H}{R} \cdot \left(Rx^4/4 - x^5/5\right)\Big|_{x=0}^{x=R}$$

$$= \frac{2\pi\rho H}{R}\, R^5\left(1/4 - 1/5\right)$$

$$= \frac{\pi\rho H R^4}{10}$$

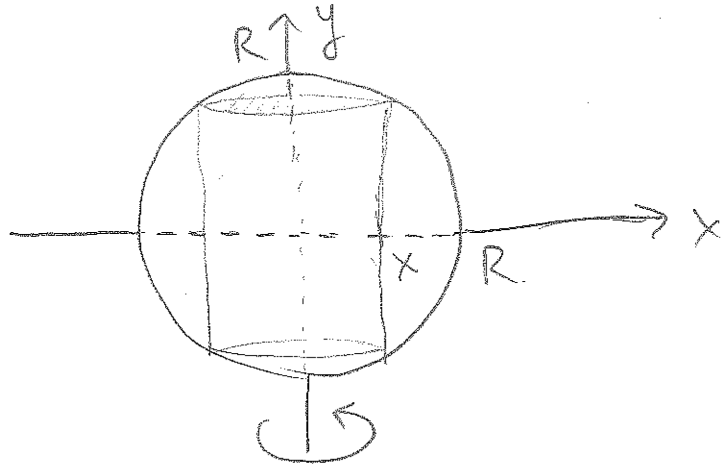
Use $M = \rho/3\, \pi R^2 H$ to get

$$I = \rho/3\, \pi R^2 H \cdot 3/10\, R^2 = \boxed{\frac{3M}{10} R^2}$$

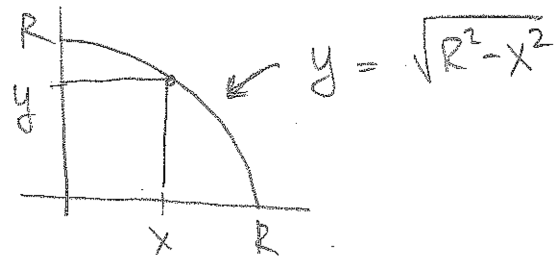
Again, try this at home but with variable density $\rho(x) = x$.

Q3 And now, a sphere of radius R, rotated about any diameter.

Ans.



At distance x from the axis, we get a cylinder of height $2y$ . . .



So, $dV = 2\pi x \sqrt{R^2 - x^2}\, dx$,

$dM = 2\pi \rho x \sqrt{R^2 - x^2}\, dx$

and, $dI = dM \cdot x^2 = 2\pi \rho x^3 \sqrt{R^2 - x^2}\, dx$.

Finally, $I = \int_0^R 2\pi\rho x^3 \sqrt{R^2 - x^2}\, dx$

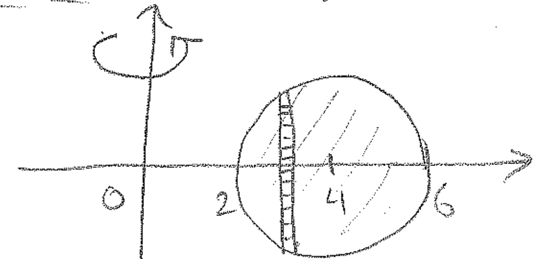
- use substitution $x = R\sin\theta$; then

$M = \frac{4\rho}{3}\pi R^3$

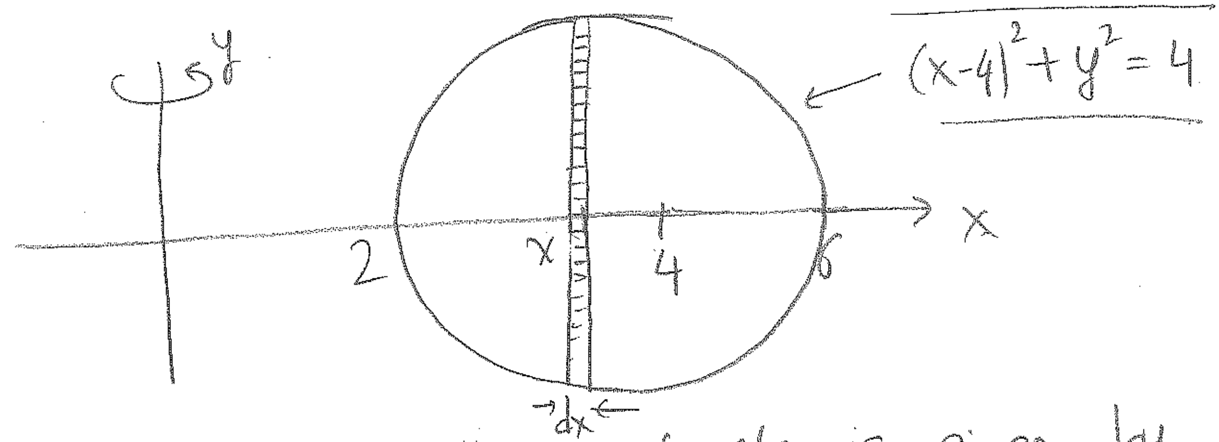
to get $I = \frac{2}{5} M R^2$.

Q4. Disk of radius 2 rotated about an axis at distance 4 from the center. Constant density ρ

Ans



At distance x from the y-axis (between 2 and 6), we have a rectangle element:



The Height of the rectangle is given by solving $(x-4)^2+y^2=4$ for y and multiplying by 2, so

$$y = \sqrt{4-(x-4)^2}, \quad \text{and so}$$

$$dA = 2\sqrt{4-(x-4)^2}\,dx,$$

and $dM = 2\rho\sqrt{4-(x-4)^2}\,dx$,

so $dI = 2\rho x^2\sqrt{4-(x-4)^2}\,dx$

Now,
$$I = \int_2^6 2\rho x^2\sqrt{4-(x-4)^2}\,dx$$
$$= 2\rho\int_2^6 x^2\sqrt{4-(x-4)^2}\,dx$$

TRIG SUB: $(x-4) = 2\sin\theta$, so $x = 4+2\sin\theta$.

and $dx = 2\cos\theta\,d\theta$.

Limits: $6-4 = 2\sin\theta$, so $\theta = \pi/2$
$2-4 = 2\sin\theta$, so $\theta = -\pi/2$.

Now,

$$I = 2\rho \int_{-\pi/2}^{\pi/2} (4+2\sin\theta)^2 \cdot 2\cos\theta \cdot (2\cos\theta\, d\theta)$$

$$= 2\rho \int_{-\pi/2}^{\pi/2} (16 + 4\sin^2\theta + 16\sin\theta) \cdot 4\cos^2\theta\, d\theta$$

$$= 2\rho \int_{-\pi/2}^{\pi/2} (64\cos^2\theta + 16\sin^2\theta\cos^2\theta\, d\theta + 64\sin\theta\cos^2\theta)\, d\theta$$

Now, $\sin\theta\cos^2\theta$ is an ODD function, and the limits $-\pi/2$ to $\pi/2$ are symmetric about 0, so

$$\int_{-\pi/2}^{\pi/2} 64\sin\theta\cos^2\theta\, d\theta = 0.$$

Thus, $$I = 2\rho \int_{-\pi/2}^{\pi/2} 64\cos^2\theta + 16\sin^2\theta\cos^2\theta\, d\theta$$

use $\boxed{\sin(2\theta) = 2\cos\theta\sin\theta}$

$$= 2\rho \int_{-\pi/2}^{\pi/2} 64\cos^2\theta + 4\sin^2(2\theta)\, d\theta.$$

We used:
$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$
$\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}$.

$$= 2\rho \int_{-\pi/2}^{\pi/2} 32\left[1+\cos(2\theta)\right] + 2\left[1-\cos(4\theta)\right] d\theta$$

$$= 2\rho \int_{-\pi/2}^{\pi/2} 30 + 32\cos(2\theta) - 2\cos(4\theta)\, d\theta$$

$$= 2\rho \left[30\theta + 16\sin(2\theta) - \underbrace{\cancel{\tfrac{1}{2}\sin(4\theta)}}_{\sin(2\pi) = \sin(-2\pi) = 0}\right]_{\theta=-\pi/2}^{\theta=\pi/2}$$

$$= 2\rho\,[30\pi + 32]$$

$$= \boxed{4\rho(15\pi + 16)}.$$ Using Mass $= \rho\cdot(\pi\cdot 2^2) = 4\pi\rho$

we get $\boxed{I = M(15 + 16/\pi)}$.