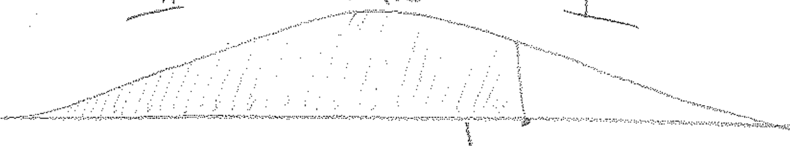
DAY 25
MON NOV 2

PROBABILITY



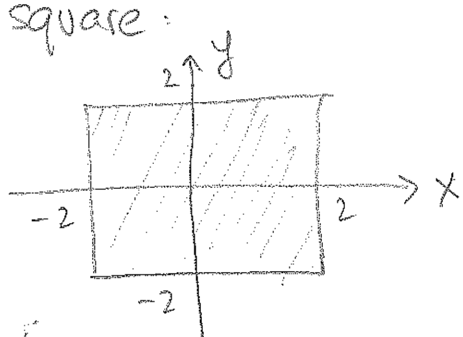
UNIFORM / FAIR PROBABILITY: Every point has equal chance of being "randomly" chosen.

Here, the computations are <u>ALL ABOUT</u> areas and volumes.

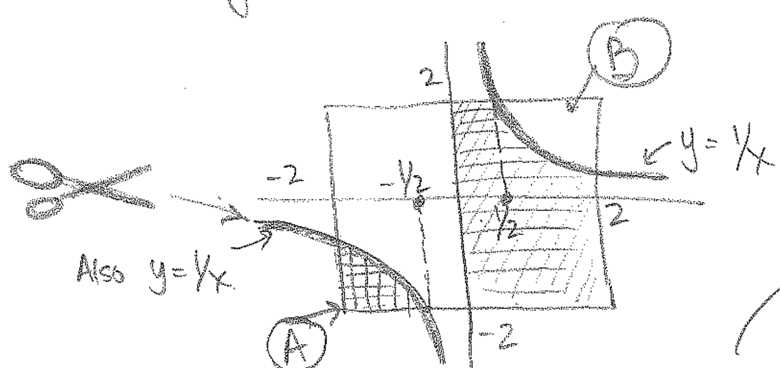
$$\mathbb{P}\begin{bmatrix}\text{uniformly random} \\ \text{point chosen from } X \\ \text{lies in region } D\end{bmatrix} = \frac{\mathrm{Vol}(X \cap D)}{\mathrm{Vol}(X)} \leftarrow \text{Intersection!}$$

Q1. Two points x and y are chosen uniformly from the region $\{z : |z| < 2\}$. What is the probability that $y \le 1/x$?

Ans. First, note that x and y are chosen uniformly from this square:



And the region we want to "hit" is shaded:



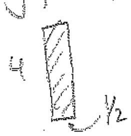
You should already <u>See</u> ⊙⊙ that the probability is $1/2$.

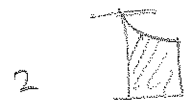
So, $$\mathbb{P}\left[y \le 1/x\right] = \frac{\text{Area of shaded region}}{\text{Area of entire square}} = \frac{1}{2}$$

(Because A = B)

(Don't panic if you don't *see* the 1/2 right away; you can always COMPUTE)

The denominator is just $4^2 = 16$, so we need to get the numerator. Clearly, we need <u>three</u> integrals:





$$\text{Area of Shaded Region} = \int_{-2}^{-1/2} \left(\tfrac{1}{x} - (-2)\right) dx + \left(4 \times \tfrac{1}{2}\right) + \int_{1/2}^{2} \left(\tfrac{1}{x} - (-2)\right) dx$$

$$= \Big(\ln|x| + 2x\Big)_{x=-2}^{x=-1/2} + 2 + \Big(\ln|x| + 2x\Big)_{x=1/2}^{x=2}$$

$$= 3 - \cancel{\ln(4)} + 2 + 3 + \cancel{\ln(4)}$$

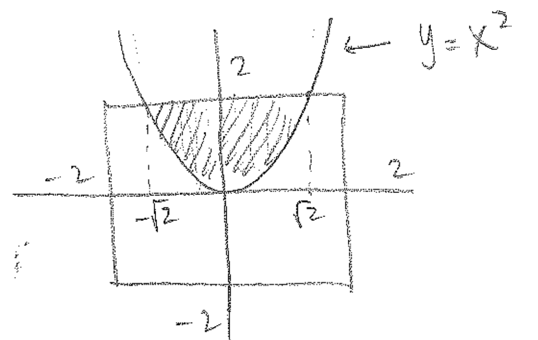
$$= 8$$

Confirmed!

So, $\text{Probability} = 8/16 = \boxed{1/2}$.

Q2 Same region as before, but find the probability that $y \geq x^2$

Ans



Total area $= 4^2 = 16$.

<u>Shaded</u> <u>area</u>: $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)\, dx$

$$= \left(2x - x^3/3\right)_{x=-\sqrt{2}}^{x=\sqrt{2}}$$

$$= \left[2\sqrt{2} - (\sqrt{2})^3/3\right] - \left[2(-\sqrt{2}) - (-\sqrt{2})^3/3\right]$$

$$= 4\sqrt{2} - 4\sqrt{2}/3 = 8\sqrt{2}/3$$

$$\text{Ans} = \frac{8\sqrt{2}/3}{16} = \sqrt{2}/6 \approx 0.236 = \underline{23.6\,\%}$$

Q3 Say you have a CYLINDER of radius R and height H. Find the radius R' so that the probability of uniformly selecting a point from the cylinder which ALSO lies in the cylinder of radius R' and height H is exactly 90%.

Ans Want to solve for R':

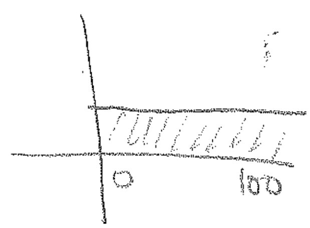
$$0.9 = \frac{\text{Volume of cylinder of radius } R', \text{ Height } H}{\text{Volume of cylinder of radius } R, \text{ Height } H}$$

So, $$0.9 = \frac{\cancel{\pi}(R')^2 \cancel{H}}{\cancel{\pi} R^2 \cancel{H}}$$

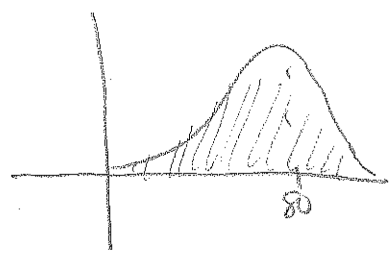
Or, $$\boxed{R' = \sqrt{0.9}\, R}$$

PROBABILITY DENSITIES

Sometimes, we WANT unfair probability. Eg, in grade distributions:



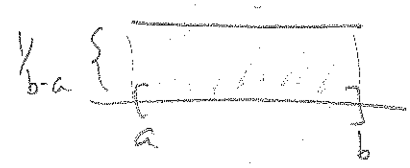
vs



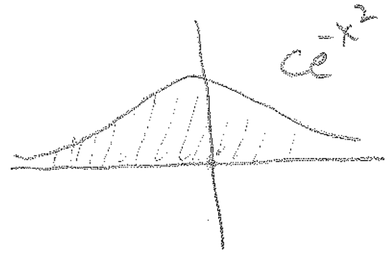
IMPORTANT Def: A PROBABILITY DENSITY FUNCTION (or PDF) on $[a,b]$ is a POSITIVE function whose integral equals 1 on its domain:

$$\underset{a \le x \le b}{p(x) \geq 0} \quad \text{AND} \quad \int_a^b p(x)\,dx = 1$$

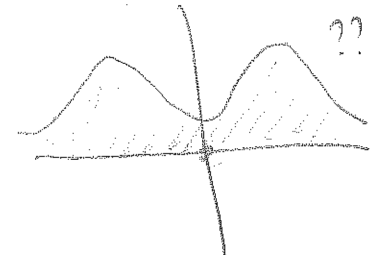
EXAMPLES :



UNIFORM



GAUSSIAN



BIMODAL.

Q4. For which constant c is $p(x) = cxe^{-x^2}$ a probability density function on $[0,\infty]$?

Ans. Well, just compute $\int_0^\infty xe^{-x^2}\,dx$. (everything is ≥ 0)

Set $u = x^2$, so $du = 2x\,dx$ and we get

$$\tfrac{1}{2}\int_0^\infty e^{-u}\,du = -\tfrac{1}{2}\left[e^{-u}\right]_0^\infty$$

$$= -\tfrac{1}{2}\left[0-1\right] = \tfrac{1}{2}.$$

So, $\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}$, meaning that if $\boxed{c=2}$,

then: $\int_0^\infty p(x)\,dx = 1$, so $p(x)$ becomes a PDF.

Q5. Assuming the PDF above, what is the probability that a randomly chosen x is:

a) ≥ 5 ? b) ≤ 20 ? c) Between 1 and 30 ?

Ans

a)	b)	c)
$\int_5^\infty 2xe^{-x^2}\,dx$	$\int_0^{20} 2xe^{-x^2}\,dx$	$\int_1^{30} 2xe^{-x^2}\,dx$