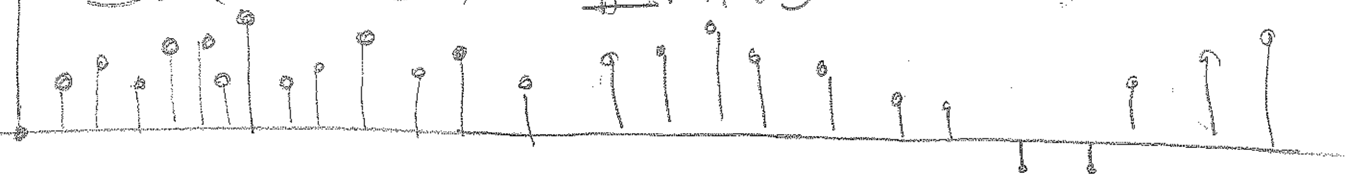
DAY 28
WED NOV 11

SEQUENCES, LIMITS



A $\boxed{\text{SEQUENCE}}$ "a" is a function $a: \mathbb{N} \to \mathbb{R}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{R}$ is the usual Real number line
(optional)
[Write $a(n)$ as a_n].

Eg:
- List: $(0, 1, 4, 9, 16, \dots)$
- Formula: $a_n = n^2$
- Recurrence: $\begin{cases} a_n = \left(\sqrt{a_{n-1}} + 1\right)^2 \\ a_0 = 0 \end{cases}$ ← (in terms of smaller $a_\cdot$'s)

LIMITS

The only USEFUL limits of sequences are at ∞:

- $\lim_{n \to \infty} a_n = L$ if for every $\varepsilon > 0$ there is N large enough so that $|a_n - L| < \varepsilon$ for all $n > N$.

Q1. Find $\lim_{n \to \infty} 1/n$.

Ans. We know the answer is zero, so let's see why.

WANT: for every $\varepsilon > 0$, a recipe for getting some N so that $|1/n - 0| < \varepsilon$ for $n > N$.
(see the definition)

i.e., WANT $1/n < \varepsilon$, or $n > 1/\varepsilon$.

So, SET $N = 1/\varepsilon$. then, $n \geq N$ forces
$|1/n - 0| < \varepsilon$, hence 0 is $\lim_{n \to \infty} a_n$

Q2 Find the value of

$$\sqrt{3-\sqrt{3-\sqrt{3-\cdots}}}$$

Ans This is $\lim_{n\to\infty} a_n$, where:

$a_0 = \sqrt{3}$, $a_1 = \sqrt{3-\sqrt{3}}$, $a_2 = \sqrt{3-\sqrt{3-\sqrt{3}}}$ etc.

Recursively, $a_n = \sqrt{3-a_{n-1}}$

So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{3-a_{n-1}}$$

Now, $\lim_{n\to\infty} a_{n-1}$ is the SAME as $\lim_{n\to\infty} a_n$, since the definition only requires some property to hold for n large enough. So, call this limit L, and note:

$$L = \sqrt{3-L}, \text{ so } L^2 = 3-L,$$

so $L^2+L-3=0$, or $L = \frac{-1\pm\sqrt{1+12}}{2}$, $\boxed{L = \frac{\sqrt{13}-1}{2}}$

(Discard negative solution, the all a's are ≥ 0).

Q3. Find $\lim_{n\to\infty} a_n$ for $a_n = \sqrt{n^2+4} - n$.

Ans Here we have $a_n = \sqrt{n^2+4} - n$, or

$$a_n = (n^2+4)^{1/2} - n.$$

Want to use binomial expansion for $n\to\infty$, so

$$a_n = n(1+4/n^2)^{1/2} - n$$

$$= n(1+O(1/n^2)) - n$$

$$= n + O(1/n) - n = O(1/n)$$

But $\lim_{n\to\infty} 1/n = 0$ by Q1, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} O(1/n) = \boxed{0}$$