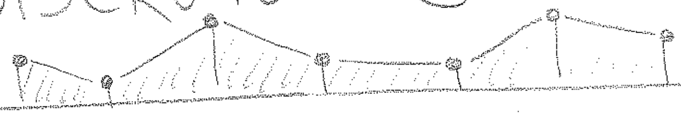
DAY 29
FRI NOV 13

DISCRETE CALCULUS



OPERATIONS ON SEQUENCES:

(Eg: $a_n = 2^{-n}$, so $(1, 1/2, 1/4, 1/8, 1/16, \dots)$)

"$\frac{d}{dx}$" 1. – Forward Difference: $(\Delta a)_n = a_{n+1} - a_n = \underline{2^{-(n+1)} - 2^{-n}}$

– Backward Difference $(\nabla a)_n = a_n - a_{n-1} = \underline{2^{-n} - 2^{-(n-1)}}$

"$\int_a^b$" 2. – Summation $\sum_{n=A}^{B} a_n = \left(1/2^A + 1/2^{A+1} + \dots + 1/2^B\right) = \dots$

3. – Shifts – $(Ea)_n = a_{n+1}$, so $(1/2, 1/4, 1/8, \dots)$

– $(E^{-1}a)_n = a_{n-1}$, so $(?, 1, 1/2, 1/4, \dots)$

$\boxed{\text{FTIC}}$:

$$\sum_{n=A}^{B} (\Delta b)_n = b_{B+1} - b_A.$$

$$\sum_{n=A}^{B} (\nabla b)_n = b_B - b_{A-1}$$

Very Important!

$\sum$: Δ or ∇

as

$\int$: d/dx.

Q1. Consider the sequence $a_n = 1/n^2$.

a) Find Δa

b) What is $\displaystyle\sum_{n=5}^{212} \frac{(2n+1)}{n^4+2n^3+n^2}$?

Ans

a) $(\Delta a)_n = a_{n+1} - a_n = \dfrac{1}{(n+1)^2} - \dfrac{1}{n^2}$

$= \dfrac{n^2 - (n+1)^2}{n^2(n+1)^2} = \boxed{\dfrac{-(2n+1)}{n^4+2n^3+n^2}}$

b) By FTIC, $\displaystyle\sum_{5}^{212} \frac{-(2n+1)}{n^4+2n^3+n^2} = \frac{1}{(213)^2} - \frac{1}{5^2}$, so

the answer is just $\boxed{1/5^2 - 1/(213)^2}$

Q2. What is $\displaystyle\sum_{n=1}^{8000} \ln(1+1/n)$?

Ans. Want to express the "summand" $\ln(1+1/n)$ as some forward or backward difference. But look:

$$\ln(1+1/n) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$$

Aha: so if we set $a_n = \ln(n)$, then:

$$\sum_{n=1}^{8000} \ln(1+1/n) = \sum_{n=1}^{8000} \Delta a_n$$

$$\text{(By FTIC)} \quad = a_{8001} - a_1$$

$$= \ln(8001) - \underbrace{\cancel{\ln(1)}}_{0}$$

FALLING POWERS

These are the "polynomials" in discrete-land.

(Hey, this looks like $k!\binom{n}{k}$ ...)

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$$

$$\Delta n^{\underline{k}} = k\, n^{\underline{k-1}}$$

and, $$\Delta^2 n^{\underline{k}} = k(k-1)\, n^{\underline{k-2}}, \text{ etc.}$$

Eg:
- $n^{\underline{2}} = n(n-1)$
- $n^{\underline{3}} = n(n-1)(n-2)$
- ⋮
- $n^{\underline{7}} = n(n-1)\cdots(n-6)$

Q3 What is: $\displaystyle\sum_{n=0}^{K} (n^2 - 3n)$ as a function of K?

Ans. Set $a_n = n^2 - 3n$. Now, remember that

$$n^{\underline{1}} = n \quad \text{and} \quad n^{\underline{2}} = n(n-1) = n^2 - n$$

So, $$n^2 = n^{\underline{2}} + n = n^{\underline{2}} + n^{\underline{1}}.$$

Thus, $$a_n = n^{\underline{2}} - 2n^{\underline{1}}$$

So, we get $$\sum_{n=0}^{k} (n^2 - 3n) = \sum_{n=0}^{K} n^{\underline{2}} - 2n^{\underline{1}}$$

SUMMARY: Write as a falling power, integrate, rewrite as ordinary power.

Now, summation becomes <u>EASY</u>, because

$$\sum_{n=A}^{B} n^{\underline{k}} = \frac{(B+1)^{\underline{k+1}} - A^{\underline{k+1}}}{(k+1)} \quad \text{(FTIC)}$$

So, $$\sum_{n=0}^{\underline{K}} n^{\underline{2}} - 2n^{\underline{1}} = \tfrac{1}{3}(\underline{K}+1)^{\underline{3}} - (\underline{K}+1)^{\underline{2}}$$ [evaluation at zero gives zero]

Finally, re-convert to polynomials:

$$\tfrac{1}{3}(\underline{K}+1)^{\underline{3}} - (\underline{K}+1)^{\underline{2}} = \tfrac{1}{3}(\underline{K}+1)(\underline{K})(\underline{K}-1) - (\underline{K}+1)(\underline{K})$$

$$= \tfrac{1}{3}(\underline{K}^3 - \underline{K}) - (\underline{K}^2 + \underline{K})$$

$$= \boxed{\tfrac{1}{3}\underline{K}^3 - \underline{K}^2 - \tfrac{4}{3}\underline{K}}.$$

So try figuring out $\sum_{n=0}^{100} (n^2 - 3n)$ by plugging $\underline{K} = 100$ into this expression.