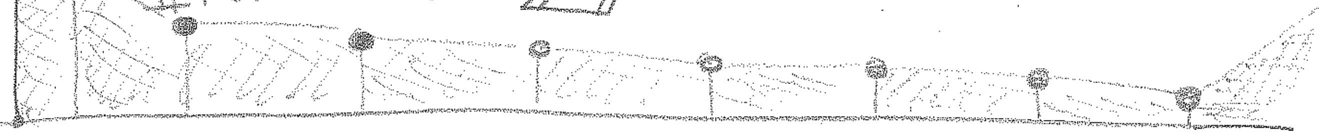
DAY 30
MON NOV 16

INFINITE ΣERIES



Tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges;

"0." n-th term test

"1." Integral / Comparison

"2." Limit Test

HARDER, MORE USEFUL!

Q1 $\sum_{n=1}^{\infty} \cos\left(\frac{1}{1+n}\right)$?

Ans. $\lim_{n\to\infty} \cos\left(\frac{1}{1+n}\right) = \cos(0) = 1 \neq 0$,

So this DIVERGES. (by n-th term test)

> <u>IMP</u>: If $\lim_{n\to\infty} a_n = 0$, we can NOT conclude that the series converges. ONE-WAY!

Q2. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Ans. The n-th term is going to zero as $n \to \infty$, so we need something else. Set $f(x) = \ln(x)/x^2$, and check

$$I = \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \ln(x)/x^2\,dx$$

By parts: $u = \ln(x)$, $dv = x^{-2}dx$
$du = dx/x$, $v = -x^{-1}$

$$\text{So } I = \underbrace{-\frac{\ln x}{x}\Big|_1^{\infty}}_{\text{finite by L'Hôpital}} + \underbrace{\int_1^{\infty} x^{-2}\,dx}_{\text{converges by p-test.}} < \infty.$$

So, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ CONVERGES by $\int$-test

[Here we do get a reverse-implication! If the $\int$ diverged, so would the Σ.]

Q3. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-4}}$

Ans. Again, set $f(x) = \frac{1}{\sqrt{x-4}}$ and examine

$$I = \int_5^{\infty} \frac{dx}{\sqrt{x-4}} = \int_5^{\infty} (x-4)^{-1/2}\, dx$$

Use u-sub, $u = x-4$, so

$$I = \int_1^{\infty} u^{-1/2}\, du,$$

diverges by integral p-test! So, $\sum_1^{\infty} \frac{1}{\sqrt{n-4}}$ also DIVERGES!

Q4. $\sum_{n=1}^{\infty} \frac{n^3}{n^5+n+1}$

Ans. Since $n^5+n+1 > n^5$ for n between 1 and ∞, note that

$$\frac{n^3}{n^5+n+1} < \frac{n^3}{n^5} = \frac{1}{n^2}.$$

But $\sum_{n=1}^{\infty} 1/n^2$ converges by p-test, so our series ALSO CONVERGES.

Q5. $\displaystyle\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-2}$

Ans. Since $n-2 < n$ (duh),

$$\frac{\sqrt{n}}{n-2} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = n^{-1/2}.$$

But $\displaystyle\sum_{n=4}^{\infty} n^{-1/2}$ diverges (by p-test), and hence so does our series.

Q6. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$

Ans. Let $a_n = n^2/n^4-1$, and compare with $b_n = 1/n^2$. Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4-1} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^4}{n^4-1} = \underline{\underline{1}}$$

Since $0 < \underline{\underline{1}} < \infty$, a_n converges if and only if b_n converges. but b_n converges by p-test!

So, Σa_n converges!

Q7. $\displaystyle\sum_{n=1}^{\infty} 1 - \cos(1/n)$

Ans. $a_n = 1 - \cos(1/n) = 1 - [1 - 1/n^2 + O(n^{-4})]$ ← Taylor!

$= 1/2n^2 + O(n^{-4})$

So, let's compare with $b_n = 1/n^2$!

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left[\frac{1}{2n^2} + O(n^{-4})\right] n^2 \quad \text{as } n\to\infty$$

$$= \lim_{n\to\infty}\left(\tfrac{1}{2} + O(n^{-2})\right) = \tfrac{1}{2} \quad \left(\text{not } \tfrac{0}{\infty}\right)$$

So, again, a_n converges because $\sum 1/n^2$ does!

MORE: use any method!

A. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2}$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+\sin(n)}$

C. $\displaystyle\sum \frac{n^2-n}{n^5+n}$

D. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$