DAY 31
FRI NOV 20

MORE CONVERGENCE TESTS

Agenda:

1. $\sqrt{\text{Root}}$ and Ratio tests
2. Absolute vs. conditional convergence

Q1. Does $\sum_{n=1}^{\infty} \left[\pi/2 - \arctan(n)\right]^n$ converge?

Ans. This is HARD to do by traditional methods (what can you compare this to?). I'd love to Taylor-expand arctan, but then we have to deal with an "n"-th power. So, ROOT TEST:

$$a_n = \left[\pi/2 - \arctan(n)\right]^n$$

So, $a_n^{1/n} = [\pi/2 - \arctan(n)$

Now, $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left[\pi/2 - \arctan(n)\right] = 0$

Since $0 < 1$, the root test guarantees convergence!

Remember when we showed $\lim_{x\to 0} (1+x)^{1/x} = e$ on Day 4? It'll be really useful now...

Q2 What about $\sum_{n=1}^{\infty} \left[1 - 1/n\right]^{n^2}$?

Ans. Even after trying to apply the root test, we have a challenging exponent: set $a_n = \left[1 - 1/n\right]^{n^2}$, and now

$$\lim_{n\to\infty} a_n^{1/n} = \left[1 - 1/n\right]^n.$$ Now what?

Well, set $b_n = \ln(a_n^{1/n}) = \ln[1 - 1/n]^n = n\ln[1-1/n]$

So,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n\ln(1-1/n)$$

This is not straightforward: n is going to ∞, but $\ln(1-1/n)$ is going to 0, so Taylor-expand.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n \cdot \left[-1/n + O(1/n^2)\right]$$

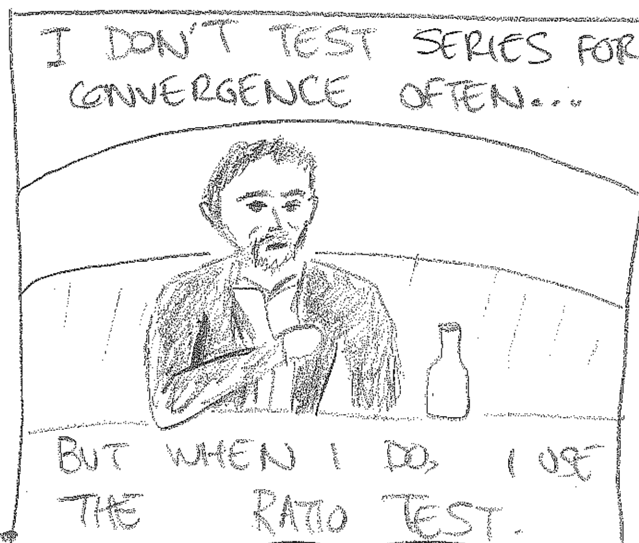
$$= \lim_{n\to\infty} \left(-1 + O(1/n)\right) = \underline{-1}.$$

Since $b_n = \ln(a_n^{1/n})$, we get

$$\lim_{n\to\infty} \ln(a^{1/n}) = -1, \text{ so } \lim_{n\to\infty} a_n^{1/n} = \underline{1/e}.$$

Since $1/e < 1$, our series <u>CONVERGES</u>!

The root test is only useful when $a_n = (\text{stuff})^n$, because taking the n^{th} root simplifies things.



<u>Ratio Test</u>:

Useful for
- factorials,
- powers;

Q3. $\sum_{n=1}^{\infty} n^k/3^n$, where k is a fixed constant.

Ans. $a_n = n^k/3^n$, so $a_{n+1} = (n+1)^k/3^{n+1}$ and the ratio (absolute value) $\rightarrow$ $|a_{n+1}/a_n| = \frac{(n+1)^k}{3^{n+1}} \cdot \frac{3^n}{n^k}$

has a MASSIVE CANCELLATION, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^k \cdot \frac{1}{3}$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^k \cdot 1/3 = \underline{\underline{1/3}} < 1$

So, CONVERGES!

Q4. What about the reciprocal? $\sum_{n=1}^{\infty} 3^n/n^k$

Ans Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^k}\cdot\frac{n^k}{3^n}$$

$$= \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^k = \underline{\underline{3}} > 1,$$

So this one <u>diverges</u>.

Q5. Try $\sum_{n=1}^{\infty} n^n/(2n)!$

Ans.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n^n}$$

$$= (n+1)\cdot\left(\frac{n+1}{n}\right)^n\cdot\frac{1}{(2n+1)(2n+2)}$$

$$= \left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{(2n+1)(2n+2)}$$

goes to "e" as $n\to\infty$

goes to "0" as $n\to\infty$.

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, giving convergence

(Now check that $\sum n^n/n!$ diverges)

Every Series $\sum a_n$

- → Converges Absolutely, i.e. $\sum |a_n|$ converges

  OR
- → Converges Conditionally, i.e. $\sum a_n$ converges but $\sum |a_n|$ does not

  OR
- → Diverges.

Q6. What does $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$ do?

Ans Set $a_n = (-1)^n/\sqrt{n}$. Immediately, we note that

$$\lim_{n\to\infty} |a_n| = 0, \quad \text{so } \sum_{n=1} a_n \text{ converges}$$

by the ALTERNATING SERIES TEST. But,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by the p-test!

So, $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$ converges CONDITIONALLY.

Q7 And, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n^2+5)}{n}$?

Ans. Again, set $a_n = (-1)^n \frac{\ln(n^2+5)}{n}$, and since the series ALTERNATES we get convergence:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\ln(n^2+5)}{n} = 0,$$

because $\ln(n^2) = 2\ln(n)$ grows much slower than n. But, note that the SERIES $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln(n^2+5)}{n}$ is term-wise bigger than $\sum_{n=1}^{\infty} 1/n$, hence it diverges.

So, the series $\sum a_n$ converges CONDITIONALLY.