DAY 32
Mon WED 23

POWER SERIES

TODAY:
Returning to Taylor Series...

A power series $\sum_{n=0}^{\infty} a_n x^n$ is:

a) A function $f(x)$ built from a sequence $\{a_n\}$
b) A family of series, one for each x-value!

Note, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f(0) = a_0$ without doing any work. So, every series of the form $\sum_{1}^{\infty} a_n x^n$ converges at $x=0$. The question is: for which OTHER x-values do we get convergence?

Q1. Find the interval of convergence for $\sum_{n=1}^{\infty} n^2 x^n$

Ans The RADIUS of convergence is $\lim_{n\to\infty} |a_n/a_{n+1}|$ so

$$R = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1 \quad \left[\text{By L'Hopital}\right]$$

So, we definitely have convergence on $(-1,1)$. To test the endpoints, first try $x=1$; get $\sum_{n=1}^{\infty} n^2$, which diverges by the "OMFG THE TERMS ARE NOT GOING TO ZERO!!"-test.

Next, try $x=-1$ and get the alternating series $\sum_{n=1}^{\infty} (-1)^n n^2$, ALSO diverges by the same test.

So: $\boxed{\text{Interval of Convergence} = (-1,1)}$

(Both are open!)

Q2 Try both

a) $\sum_{n=1}^{\infty} (-1)^{n+1} x^n/n$ b) $\sum_{n=0}^{\infty} (-1)^{n+1} x^{2n+1}/2n+1$

Ans No need to panic! a) is $\ln(1+x)$ and b) is arctan(x). Remember those ??

a)
$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n-1} \cdot 1/n}{(-1)^n \cdot 1/n+1} \right|$$
$$= \lim_{n\to\infty} \frac{n+1}{n} = \underline{1}.$$

So, need to test endpoints of (-1,1). At $x=-1$, get $\sum_{n=1}^{\infty} (-1)^n/n$, the ALTERNATING HARMONIC SERIES, which converges! But at $x=1$, we get the ACTUAL HARMONIC series, which diverges; so, the interval of convergence = [-1, 1)

b) Similarly as in a),
$$R = \lim_{n\to\infty} \left| \frac{2(n+1)+1}{2n+1} \right| = \lim_{n\to\infty} \frac{2n+3}{2n+1} = \underline{1}.$$

Again, test endpoints:

@ $\underset{x}{+1}$, get $\sum_{n=0}^{\infty} (-1)^{n+1}/2n+1$, $\left\{\begin{matrix}\text{alternating series} \\ \text{with terms} \to 0\end{matrix}\right\}$, converges!

@ $\underset{x}{-1}$, get $\sum_{n=0}^{\infty} (-1)^{3n+2}/2n+1$, { same }, converges!

So, interval of convergence = [-1, 1]

Q3. And now, $\sum_{n=0}^{\infty} (2x)^n/n!$

Ans (This is just e^{2x} ..), $a_n = 2^n/n!$, so
$$R = \lim_{n\to\infty} \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2} = \underline{\infty}$$

So, converges on $I = (-\infty, \infty)$, i.e. EVERYWHERE!

MORE GENERALLY (i.e, when Taylor expanding about points different from 0), we deal with $\sum_{n=0}^{\infty} a_n (x-c)^n$.

Q4.	$\sum_{n=0}^{\infty} (x+5)^n \cdot n!$
Ans.	If $y=(x+5)$, then we are back to the usual case for $\sum_{n=0}^{\infty} n! \cdot y^n$. Now, $R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = \underline{\underline{0}}$. So, the series <u>ONLY</u> converges at $y=0$, or $x=-5$
Q5.	$\sum_{n=0}^{\infty} \frac{(-2)^n \cdot (x+1)^n}{n}$
	Again, set $y = x+1$ and look at $\sum_{n=0}^{\infty} \frac{(-2)^n}{n} y^n$. So, $R = \lim_{n\to\infty} \left\| \frac{(-2)^n}{n} \cdot \frac{n+1}{(-2)^{n+1}} \right\| = \lim_{n\to\infty} \frac{n+1}{2n} = 1/2$, so $\begin{cases} \overset{(x+1)}{\overset{\shortparallel}{y}} \text{ in } (-1/2, 1/2) \\ \Rightarrow x \text{ in } (-3/2, -1/2) \end{cases}$ Try endpoints: · at $y = 1/2$, get $\sum_{n=0}^{\infty} \frac{(-2)^n \cdot (1/2)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ <u>converges</u> by alternating series test. · at $y=-1/2$, get $\sum_{n=0}^{\infty} \frac{(-2)^n \cdot (-1/2)^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$, <u>diverges</u> by p-test So, $\left[\begin{array}{c}\text{Interval of} \\ \text{convergence}\end{array}\right]$ is $(-1/2, 1/2]$ for $y = x+1$, so $\boxed{(-3/2, 1/2]}$ for x.

<u>WITHIN</u> the interval of convergence, we can treat power series as polynomials! Differentiating, integrating etc. is allowed term-by-term.

Q6 Which function has the Taylor series

$$1 + 2x + 3x^2 + 4x^3 + \cdots ?$$

Ans Want $f(x) = 1 + 2x + 3x^2 + \cdots$, but look:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1,$$

so $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + \cdots \quad \text{for } |x| < 1$$

So, $$\boxed{\frac{1}{(1-x)^2}} = 1 + 2x + 3x^2 + \cdots \quad \text{for } |x| < 1$$

Answer!

<u>WARNING</u>: Almost ANYTHING you do with power series OUTSIDE the interval of convergence is WRONG.

<u>Eg</u>. $$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

If we "forget" $|x| < 1$, and plug $x = 2$:

$$\frac{1}{1-2} = 1 + 2 + 4 + \cdots$$

So, $$\underbrace{-1}_{\text{negative}} = \underbrace{1 + 2 + 4 + \cdots}_{\text{positive}}$$

<u>GARBAGE!</u>