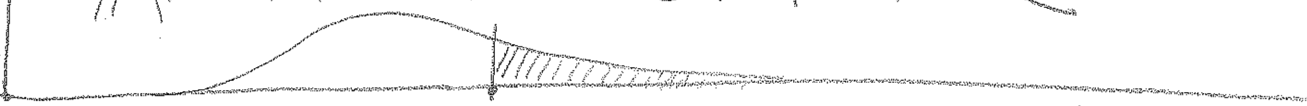
DAY 33
WED NOV 25

APPROXIMATION & ERROR



- How many terms of your power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ do you need to get an approximation with controlled error?

- We can decompose this power series:

$$\underbrace{f(x)}_{\text{(infinite sum)}} = \underbrace{\sum_{n=0}^{N} a_n x^n}_{\text{(finite sum)}} + \underbrace{E_N(x)}_{\text{(Error), or "Remainder"}}$$

some "large" N

where

Trinity

ALTERNATING — a) $|E_N(x)| \leq |a_{N+1}|$ ONLY if the series is alternating.

INTEGRAL — b) $|E_N(x)| \leq \int_N^{\infty} a(x)\,dx$ always.

TAYLOR — c) $|E_N(x)| \leq \frac{C}{(N+1)!}|x|^{N+1}$, where C is the max value of $|f^{(n+1)}(t)|$ for $0 \leq t \leq x$ [(n+1)st derivative].

Q1. How many terms in the expansion of $\ln(1+x)$ do we need so that the error is bounded by 0.001?

Ans

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \qquad \text{for } |x| < 1.$$

So, the n^{th} term here is $a_n = (-1)^{n+1}/n$. Note that the series ALTERNATES, so for $|x| \leq 1$,

$$|E_N(x)| \leq |a_{N+1}| = \underline{\underline{1/N+1}}$$

To guarantee $|E_N(x)| \le 0.001$, we just solve for "N" in

$$\frac{1}{N+1} \le 0.001$$

So, $N+1 \geqslant \frac{1}{0.001} = 1000$

$$\Rightarrow \boxed{N \geqslant 999}$$

So, we need at least 999 terms to know for sure that the error is ≤ 0.001

Q2 How many terms in the Taylor expansion of e^{3x} do we need to guarantee an error ≤ 0.02 in the interval $[0,5]$?

Ans

$$e^{3x} = \sum_{n=0}^{\infty} (3x)^n/n! \quad \text{for all } x,$$

So the n^{th} term $a_n = 3^n/n!$ Note that (sadly) this series is NOT alternating, so we can't do what we did before. I'm also unwilling to use the integral $\int_N^\infty a(x)\,dx$ because $a(x) = 3^x/x!$, whatever that means. So, we will use Taylor's bound:

$$|E_N(x)| \le \frac{C}{(N+1)!}|x|^{N+1} \quad \text{for } 0 \le x \le 5$$

where $C = \begin{pmatrix} \text{max of the } (N+1)\text{-st} \\ \text{derivative of } e^{3t} \text{ on} \\ \text{the interval } [0,5] \end{pmatrix}$ $\left(\begin{matrix}\text{max of } e^{3t} \text{ on} \\ 0 \le t \le 5 \text{ is } e^{3\times 5}\end{matrix}\right)$.

$\downarrow$

Well, $\left(\frac{d}{dt}\right)^{N+1}(e^{3t}) = 3^{N+1} \cdot e^{3t} \le \underline{3^{N+1} \cdot e^{15}}$ for $0 \le x \le 5$

So, $C = 3^{N+1} \cdot e^{15}$, and we get

$$|E_N(x)| \leq \frac{3^{N+1} \cdot e^{15}}{(N+1)!} \cdot |x|^{N+1} \quad \text{for } 0 \leq x \leq 5$$

$$\leq \frac{3^{N+1} \cdot e^{15}}{(N+1)!} \cdot 5^{N+1} \quad (\text{because } x \leq 5)$$

$$= \frac{15^{N+1} \cdot e^{15}}{(N+1)!}$$

So, we want <u>any N</u> for which $\frac{15^{N+1} \cdot e^{15}}{(N+1)!} \leq 0.02$

Q3. And now, how many terms of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2n^2+1}$ to get error ≤ 0.01?

Ans. Well, it's not alternating, but I think we have a nice shot at using the Integral bound. So,

$$|E_N(x)| \leq \int_N^{\infty} \frac{dt}{2t^2+1}$$

Hey! How do we integrate $\int \frac{dt}{2t^2+1}$ again? Well, set $t = \tan\theta/\sqrt{2}$, so $2t^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Now, $dt = \sec^2\theta/\sqrt{2}\, d\theta$, so

$$\int \frac{dt}{2t^2+1} = \int \frac{\cancel{\sec^2\theta}}{\sqrt{2}\,\cancel{\sec^2\theta}}\, d\theta = \frac{\theta}{\sqrt{2}}$$

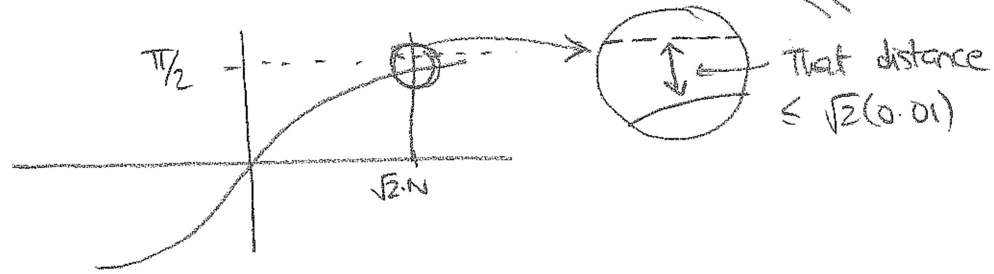
$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}t) \quad \left(\text{because } t = \frac{\tan\theta}{\sqrt{2}}\right)$$

So, $|E_N(x)| \leq \frac{1}{\sqrt{2}} \arctan(\sqrt{2}t) \Big|_{t=N}^{t=\infty} = \frac{1}{\sqrt{2}} \left[\pi/2 - \arctan(\sqrt{2}N) \right]$

So, we want N large enough to satisfy

$$\frac{1}{\sqrt{2}}\left[\pi/2 - \arctan(\sqrt{2}N)\right] \leq 0.01$$

or, $\arctan(\sqrt{2}N) \geqslant \pi/2 - \sqrt{2}(0.01)$



DISCUSSION:

Which bound should we use to control errors in

a) $\sum (-1)^n / \sqrt[3]{n}$? (Alternating)

b). $\sum 1/n^3$ (Integral)

c) $\cosh(x) = \sum x^{2n}/(2n)!$ (Taylor)

d) $\frac{1}{1-x} = \sum x^n$ (Taylor)
