QUIZ 1 SOLUTIONS

Problem 1

(**15** Points) What is the Taylor series of $f(x) = \frac{1}{1-\arctan(2x)}$ including all terms with order $\leqslant 3$?

Ans: At x near 0, we have (for $|\arctan(2x)| < 1$) the geometric series:

$$\frac{1}{1-\arctan(2x)} = 1 + \arctan(2x) + \arctan^2(2x) + \arctan^3(2x) + O(\arctan^4(x)),$$

and also that

$$\arctan(2x) = 2x - \frac{(2x)^3}{3} + O(x^5)$$

Plugging the second expression into the first, we have

$$\frac{1}{1-\arctan(2x)} = 1 + \left(2x - \frac{(2x)^3}{3} + O(x^5)\right) + \left(2x - \frac{(2x)^3}{3} + O(x^5)\right)^2 + \left(2x - \frac{(2x)^3}{3} + O(x^5)\right)^3 + O(x^4)$$

Collecting the terms of order three and below, we obtain

$$\boxed{\arctan(2x) = 1 + 2x + 4x^2 + \frac{16}{3}x^3 + O(x^4)}$$

Problem 2

(**5** Points) For which values of x does the Taylor series for $f(x) = \frac{1}{x}$ about $x = 1$ converge? You *don't* need to compute terms, just figure out the interval!

Ans: We know that the Taylor series of $\frac{1}{1-y}$ at $y = 0$ converges whenever $|y| < 1$, so let's rewrite:

$$\frac{1}{x} = \frac{1}{1-(1-x)},$$

and conclude that we have convergence whenever $|1-x| < 1$, and hence for $\boxed{0 < x < 2}$.

Problem 3

(**10** Points) What is $\lim_{x\to a} \frac{x-a}{\ln x - \ln a}$ for a constant $a > 0$?

Ans: Taylor series will not be too helpful here since we'd have to expand $\ln(x)$ about $a > 0$ rather than expanding $\ln(1+x)$ near $x = 0$. So we use l'Hôpital's rule instead, checking first that $\frac{x-a}{\ln x - \ln a}$ has the familiar $0/0$ form at $x = a$. Differentiating both numerator and denominator with respect to x, we obtain

$$\lim_{x\to a} \frac{x-a}{\ln x - \ln a} = \lim_{x\to a} \frac{1}{1/x} = \boxed{a}.$$

Problem 4

(**10** Points) Find the *largest* integer n so that $\sin(x) - \arctan(x)$ is $O(x^n)$ as $x \to 0$.

Ans: Just Taylor-expand both functions near 0:

$$\sin(x) = x + \frac{x^3}{6} + O(x^5) \text{ and } \arctan(x) = x - \frac{x^3}{3} + O(x^5),$$

so $\sin(x) - \arctan(x) = \frac{x^3}{6} + O(x^5)$. Now, let's examine the limit of $\frac{\sin(x)-\arctan(x)}{x^n}$ as $x \to 0$ and see which n-value guarantees finiteness:

$$\lim_{x\to 0} \frac{\sin(x) - \arctan(x)}{x^n} = \lim_{x\to 0} \frac{\frac{x^3}{6} + O(x^5)}{x^n} = \lim_{x\to 0} \frac{x^{3-n}}{6} + O(x^{5-n}),$$

which is finite only for $n \leqslant 3$, so the largest possible n is $\boxed{3}$.

Problem 5

(**10** Points) What is the Taylor series (include all terms of order 2 and below) of $f(x) = x^{1/2}$ near $x = 4$? What is your best guess for the value of $\sqrt{4.4}$ obtained by using only the linear part (i.e., terms of order 0 and 1 only) of this series?

Ans: The Taylor series of $f(x)$ at $x = a$ has the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + O\left((x-a)^3\right).$$

Clearly, $f(4) = 2$, so we only need to compute the first and second derivatives of $f(x) = x^{1/2}$ and evaluate them at 4. First, note that

$$f'(x) = \frac{1}{2}x^{-1/2}, \text{ so } f'(4) = \frac{1}{4}.$$

Next, we have

$$f''(x) = -\frac{1}{4}x^{-3/2}, \text{ so } f''(4) = -\frac{1}{32}.$$

Therefore, the desired Taylor series is

$$\boxed{f(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + O\left((x-4)^3\right).}$$

Note that the linear part is just $2 + \frac{1}{4}(x-4)$, so it evaluates to $\boxed{2.1}$ at $x = 4.4$.