(PRACTICE) QUIZ 2

Instructions

Please answer the following questions to the best of your ability and understanding **within 30 minutes**. Do not use books, notes, the internet, calculators, etc. You might find the following information useful:

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \int \cot(x)dx = \ln(\sin(x)) + C$$

Problem 1

(**10** Points) Evaluate the definite integral $\int_0^{\pi/2} x^2 \sin(2x)dx$ using a suitable integration technique.

Ans We should use integration by parts here. To start, set $u = x^2$ and $dv = \sin(2x)dx$, so $du = 2xdx$ and $v = -\frac{1}{2}\cos(2x)$. Then,

$$\int x^2 \sin(2x)dx = -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)dx$$

We now need to solve $\int x\cos(2x)dx$, again by parts. This time, we use $u = x$ and $dv = \cos(2x)dx$, so $du = dx$ and $v = \frac{1}{2}\sin(2x)$. Then,

$$\int x\cos(2x)dx = \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)dx = \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x) + C$$

Plugging this back into our earlier integration by parts expression, we have

$$\int x^2\sin(2x)dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x) + C$$

We're not done yet! Remember, the original problem involved limits (it was a definite integral) and we've only found the antiderivative. We still need to evaluate it between $x = 0$ and $x = \frac{\pi}{2}$. Since $\sin(0) = 0 = \sin(\pi)$ and $-\cos(\pi) = 1 = \cos(0)$, we have the final answer:

$$\boxed{\int_0^{\pi/2} x^2\sin(2x)dx = \frac{\pi^2}{8} - \frac{1}{2}}$$

Problem 2

(**10** Points) If $\frac{dx}{dt} = 3x$, at which value of t will x equal 4 times its initial value?

Ans As we've seen a billion times already, the solution to this differential equation is

$$x(t) = x(0)e^{3t},$$

and we only need to find the value of t at which $x(t) = 4x(0)$. So plug in and solve:

$$4x(0) = x(0)e^{3t}, \text{ so } e^{3t} = 4, \text{ so } \boxed{t = \frac{\ln(4)}{3}}.$$

If for some reason we forget the solution of $\frac{dx}{dt} = ax$, note that it is quite easy to derive: just rearrange to $\frac{dx}{x} = adt$, integrate both sides, and use the exponential function to kill of that natural log on the left!

Problem 3

(**10** Points) Does the improper integral $\int_{-1}^{\infty} \frac{dx}{\sqrt{x^3+2}}$ converge or diverge? Carefully explain why.

Ans The only place where our integrand has a singularity is when $x^3 = -2$, or at $x = -2^{1/3}$. Fortunately, the domain of integration does not include the negative cube root of 2 (because that value, whatever it is, must be more negative than -1). So, the only trouble here is that one of the limits of integration is $+\infty$, and we will have to use a p-test of Type B.
To put this integral in a form that resembles a p-integral, first perform the substitution $u = x^3 + 2$, so we have $x = (u-2)^{1/3}$ Also, $du = 3x^2 dx = 3(u-2)^{2/3} dx$. The lower limit $x = -1$ after substitution becomes $(-1)^3 + 2 = 1$ and the upper limit remains at $+\infty$. Putting all this together, our integral equals

$$\int_1^{\infty} \frac{1}{3} u^{-1/2} (u-2)^{-2/3} du$$

This does not look like the integral of $u^{-p} du$, so we must approximate it. Examining the integrand as $u \to \infty$ (without that useless $\frac{1}{3}$ scaling), we can pull out a power of u from the second factor:

$$u^{-1/2}(u-2)^{-2/3} = u^{-1/2}\left[u\left(1-\frac{2}{u}\right)\right]^{-2/3} = u^{-1/2}u^{-2/3}\left(1-\frac{2}{u}\right)^{-2/3}$$

Combining the powers of u, we get

$$u^{-(1/2+2/3)}\left(1-\frac{2}{u}\right)^{-2/3} = u^{-7/6}\left(1+O(u^{-1})\right)$$

where the big-O approximation at the end comes from the binomial expansion, which we can use because $\frac{2}{u}$ gets small when u gets large. Thus, the integrand is $u^{-7/6}(1+O(u^{-1}))$, which approximately reduces to the type B p-integral with $p = 7/6 > 1$, so we have convergence.

Problem 4

(**10** Points) Consider the linear ODE $\frac{dy}{dx} = y\cot(x) + \sin^3(x)$.

Part A. Find the integrating factor.

Ans For $\frac{dy}{dx} = A(x)y + B(x)$, the integrating factor $I(x)$ is given by

$$I(x) = e^{-\int A(x)dx}$$

In our case, $A(x) = \cot(x)$, so

$$I(x) = e^{-\int \cot(x)dx} = e^{-\ln(\sin(x))} = \frac{1}{\sin(x)} = \boxed{\csc(x)}.$$

You are encouraged to refrain from committing seppuku if you did not already know the integral of cotangent: note that it was provided on the first page of the Quiz.

Part B. Find the general solution to this ODE.

Ans The general solution is given by

$$y(x) = \frac{1}{I(x)}\int I(x)B(x)dx,$$

where I is the integrating factor $I(x) = \csc(x)$ from the previous part and $B(x) = \sin^3(x)$. So, we have

$$y(x) = \sin(x)\int \csc(x)\sin^3(x)dx = \sin(x)\int \sin^2(x)dx.$$

I hope everyone can integrate $\sin^2(x)$ using a suitable identity involving $\cos(2x)$ from the first page of this Quiz.

Problem 5

(**10** Points) Consider the ODE $\frac{dy}{dx} = (e^x - 1)(x^2 - 2)$.

Part A. Find all the equilibria.

Ans The equilibria occur when the right side equals 0, so either $e^x = 1$ or $x^2 = 2$. Thus, the equilibria are $\{-\sqrt{2}, 0, \sqrt{2}\}$.

Part B. Classify each equilibrium as stable or unstable.

Ans One could either evaluate the derivative of $(e^x - 1)(x^2 - 2)$ at each of the three equilibria mentioned above, or one could plug in intermediate values. I prefer the second method: let's use $x = 1$, and note that $(e - 1)(1 - 2) < 0$, so $\frac{dx}{dt}$ is decreasing in the region between 0 and $\sqrt{2}$. Similarly plugging in other values (like -1, ± 4, etc) we get:

$$-- < -- (-\sqrt{2}) -- > -- (0) -- < -- (\sqrt{2}) -- > --$$

Part C. What is $\lim_{t\to\infty} x(t)$ if $x(0) = -1$?

Ans Since -1 lies between $-\sqrt{2}$ and 0, and since x is increasing with t in this interval (see the line above), the limit must equal 0 as $t \to \infty$.

Part D. What is $\lim_{t\to-\infty} x(t)$ if $x(0) = 7$?

Ans For the same reason as in the previous answer, if $x(0) = 7$ then as $t \to -\infty$ then $x(t) \to \sqrt{2}$.