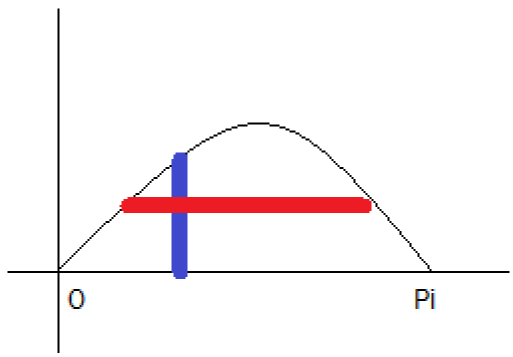
# QUIZ 3 SOLUTIONS

## Problem 1

(**15** Points) Let $R$ be the region obtained by rotating the graph of $y = \sin^2(x)$ for $0 \leqslant x \leqslant \pi$ about the $y$-axis. What is the volume of $R$? (**Hint**: a $dx$ integral will be nicer than a $dy$ integral)

**Answer**: Here's a picture of what we're up against: a $dx$ integral involves slicing across the blue line while a $dy$ integral involves slicing across the red line:




The reason I suggested that you chop along blue rather than red is simple: we know that the height of the blue slice above each $x$ is $\sin^2(x)$, whereas if you wanted to figure out the width of the red slice at height $y$, then you'd have to solve $\sin^2(x) = y$ for $x$ in terms of $y$. So, we chop along the blue lines, and note that the area element involves a cylinder of radius $x$ and height $\sin^2(x)$. So, the area element is $dA = 2\pi x \sin^2(x)dx$, and the limits clearly run from$0$ to $\pi$. Finally, we have an integral which computes the desired area:

$$A = 2\pi \int_0^\pi x \sin^2(x)dx$$

As usual, we immediately replace the $\sin^2(x)$ by $\frac{1-\cos(2x)}{2}$, which leaves

$$A = \pi \int_0^\pi (x - x\cos(2x))dx = \pi \int_0^\pi x dx - \pi \int_0^\pi x\cos(2x)dx.$$

The first integral is very straightforward, and evaluates to $\frac{\pi^3}{2}$, so we attack the second integral via integration by parts. Set $u = x$ and $dv = \cos(2x)dx$ so that $du = dx$ and $v = \frac{1}{2}\sin(2x)$. Now,

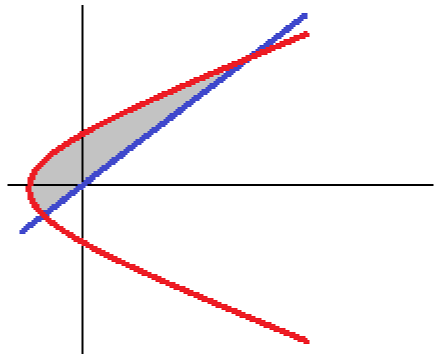
$$A = \frac{\pi^3}{2} - \frac{x}{2}\sin(2x)\Big|_{x=0}^{x=\pi} - \frac{1}{2}\int_0^\pi \sin(2x)dx.$$

The middle term and final integral both evaluate to zero, so in fact $\boxed{A = \frac{\pi^3}{2}}$.

## Problem 2

(**10** Points) Find the area of the region contained between the graphs of $x = y^2 - 2$ and $x = y$.

**Answer**: I really hope you drew a picture for this one. Behold: we want the area of the gray shaded region. The blue line is $y = x$ and the red parabola is $x = y^2 - 2$ (it touches the $x$-axis at $-2$).



We need to figure out the two points where red intersects blue, but even before that, note that we want a $dy$ integral rather than a $dx$ integral, because chopping along the $x$-axis would require two integrals. Now, to figure out the points of intersection, we must solve

$$y = y^2 - 2, \text{ so } y^2 - y - 2 = 0, \text{ or } (y-2)(y+1) = 0$$

So, $y$ runs from $-1$ to 2. At each such $y$-value, our shaded region is bounded on the left by $x = y^2 - 2$ and on the right by $x = y$. So, the area element is $dA = (y - (y^2 - 2))dy$, and our area is now computable by a single integral:

$$\begin{aligned} A &= \int_{-1}^{2} (y - y^2 + 2)\,dy \\ &= \left(\frac{y^2}{2} - \frac{y^3}{3} + 2y\right)\Big|_{y=-1}^{y=2} \\ &= \left(\frac{4}{2} - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) \\ &= \boxed{\frac{9}{2}} \end{aligned}$$

## Problem 3

(**15** Points) Use polar coordinates to find the area contained *inside* the circle of radius 1 centered at $(1, 0)$ but *outside* the circle of radius 1 centered at $(0, 0)$.

**Answer**: Picture! We want the gray region in the diagram below: it is outside the blue circle (radius 1, centered at the origin) and inside the red circle (radius 1, centered at $(1, 0)$):

includegraphicsProb3.png

We must determine the $\theta$-coordinates of the intersection points $A$ and $B$ shown above. The first circle has cartesian equation $x^2 + y^2 = 1$, which is the polar equation $r = 1$. The second circle has equation $(x-1)^2 + y^2 = 1$, which becomes $r = 2\cos(\theta)$. Setting these two equal, we have $1 = 2\cos(\theta)$, which means that $\theta = \pm\frac{\pi}{3}$ are the coordinates of $A$ and $B$.
For each such $\theta$ value, the area element is a difference of two triangular wedges coming out from the origin: the first one terminates at the blue curve and the second one at the red curve. Their difference has area $dA = \frac{1}{2}4\cos^2(\theta) - 1)d\theta$. So,

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4\cos^2(\theta) - 1)\,d\theta.$$

Now, use $4\cos^2(\theta) = 2(1+\cos(2\theta))$ to obtain

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos(2\theta)+1)d\theta$$
$$= \frac{1}{2}(\sin(2\theta)+\theta)\Big|_{\theta=-\pi/3}^{\theta=\pi/3}$$

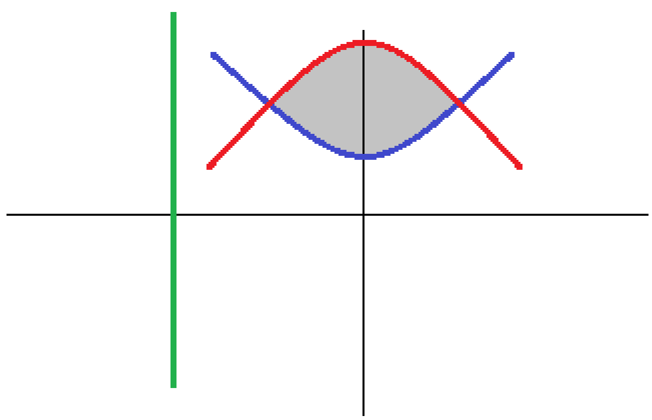
We are evaluating an odd function in a symmetric domain, so we can cancel the leading $\frac{1}{2}$ and just evaluate at the upper limit of $\frac{\pi}{3}$. This gives $A = \sin(2\pi/3) + \pi/3 = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$.

## Problem 4

(**10** Points) Let $A$ be the region contained above $y = x^2+1$ but below $y = 2-x^2$. **Set up, but do not solve** an integral which computes the volume of the solid obtained by rotating $A$ about the line $x = -1$.

Note: an earlier version of the solutions answered a different question where $A$ was rotated around $y = -1$ rather than $x = -1$. Here is the correct answer.

**Answer**: Here's the picture: $y = x^2+1$ is the blue curve, $y = 2-x^2$ is red, the line $x = -1$ is green, and the region we want to rotate about it is highlighted gray:



To get the intersection points, solve $x^2+1 = 2-x^2$ and get $x = \pm\frac{1}{\sqrt{2}}$. Now you could set this up as either a $dx$ integral or a $dy$ integral, but$dx$ is much, much simpler in this case. Note that over each $x$ in $\left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, the cross-sectional region is a line-segment of height red-blue, i.e., $(2-x^2)-(x^2+1) = (1-2x^2)$. When you rotate this segment about $x = -1$, you get a cylinder of radius $x+2$ and height $(1-2x^2)$, so the volume element is $dV = 2\pi(x+2)(1-2x^2)dx$, and we have

$$\boxed{V = 2\pi\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (x+2)(1-2x^2)dx}.$$