QUIZ 4

Instructions

Please answer the following questions to the best of your ability and understanding **within 30 minutes**. Do not use books, notes, the internet, calculators, etc.

Problem 1

(**10** Points) Consider the sequence $a_n = \left(\frac{n}{n+2}\right)^n$.

Part A. (6 Points) Either compute $\lim_{n\to\infty} a_n$, or explain why this sequence diverges.

Answer: Set $b_n = \ln(a_n) = \ln\left(\frac{n}{n+2}\right)^n = n\ln\left(\frac{n}{n+2}\right)$. Now rearrange the stuff inside the logarithm a bit, writing the numerator as $n = n+2-2$ so that we have $b_n = \ln\left(\frac{1}{-}\frac{2}{n}\right)$, so we can use the Taylor expansion for natural log (because $\frac{2}{n} \to 0$ as $n \to \infty$). Therefore,

$$b_n = n\ln\left(1 - \frac{2}{n}\right) = n\left(-\frac{2}{n} + O(n^{-2})\right) = -2 + O(n^{-1})$$

Therefore, $\lim_{n\to\infty} b_n = -2$ and since $b_n = \ln(a_n)$ we have $\lim_{n\to\infty} a_n = e^{-2}$. So the sequence converges to $\frac{1}{e^2}$.

Part B. (4 Points) Does the series $\sum_{n=0}^{\infty} a_n$ converge or diverge? Explain why.

Answer: Since the terms a_n comprise a sequence whose limit is not zero, the series diverges by the n-th term test.

Problem 2

(**15** Points) Carefully explain whether the following series converge or diverge, making sure that you mention which covergence test(s) have been used.

Part A. (5 Points) $\sum_{n=1}^{\infty} n^2\left(e^{-1/n^3} - 1\right)$

Answer Note that $e^{-1/n^3} - 1 = -\frac{1}{n^3} + O(n^{-6})$, so the summand $n^2 e^{-1/n^3}$ equals $\frac{-1}{n} + O(n^{-4})$. Note that the leading term is always negative (and not alternating). Therefore, this series diverges by limit comparison to the (negative) harmonic series $-\sum \frac{1}{n}$.

Part B. (5 Points) $\sum_{n=1}^{\infty}(-1)^n\left[\left(\frac{n}{n+2}\right)^n - 1\right]$ (**Hint**: it will help if you have solved Problem 1 first).

Answer: This is an alternating series, but the n-th term $\left(\frac{n}{n+2} - 1\right)^n$ does *not* go to zero as $n \to \infty$: it goes to e^{-2} by the previous question. Therefore, this series diverges by the alternating series test.

Part C. (5 Points) $\sum_{n=1}^{\infty} \frac{2^n \ln(n)}{(2n)!}$

Answer: Ratio test! Here $a_n = \frac{2^n \ln(n)}{(2n)!}$, so

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2^{n+1}\ln(n+1)}{(2n+2)!} \cdot \frac{(2n)!}{2^n \ln(n)}.$$

Many things will cancel:

$$\rho = \lim_{n\to\infty} 2 \cdot \left(\frac{\ln(n+1)}{\ln(n)}\right) \frac{1}{(2n+1)(2n+2)}$$

It should be clear that $\rho = 0$ for the following reasons. First, 2 is a constant so who cares? Second, that ratio of natural logs limits to 1: to see this, expand out the numerator and use the identity $\ln(ab) = \ln(a) + \ln(b)$:

$$\ln(n+1) = \ln(n(1+1/n)) = \ln(n) + \ln(1+1/n),$$

so when you divide this by $\ln(n)$ you get $1+$ something going to 0. The last factor contains a quadratic expression in the denominator with numerator = 1, so that certainly goes to zero for large n. Since $\rho = 0 < 1$, this series converges by the ratio test.

Problem 3

(**15** Points) Consider the power series $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{1-3n^2}$.

Part A. (8 Points) Find the interval of convergence.

Answer: The radius of convergence is given by $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, so

$$R = \lim_{n\to\infty} \frac{3(n+1)^2 - 1}{3n^2 - 1}$$

(Note that because of the absolute value I have replaced the negative expressions like $1 - 3n^2$ by positive ones like $3n^2 - 1$). This limit equals 1 by L'Hôpital or by comparison of leading terms in numerator and denominator (both are $3n^2$). So, the series definitely converges on $(-1, 1)$ and we only need to check our endpoints. At $x = 1$ we have an alternating series $\sum \frac{(-1)^n}{1-3n^2}$, which converges by the alternating series test: the terms are going to zero because of the $3n^2$ in the denominator. At $x = -1$, you get a non-alternating series $\sum \frac{1}{1-3n^2}$, which also converges by comparison to $\frac{1}{n^2}$. So, the interval of convergence is $[-1, 1]$.

Part B. (7 Points) Use any convenient method to find a suitable N so that the error when approximating $f(x)$ by the first N terms of its power series is guaranteed to be smaller than 0.01.

Answer: The series is alternating, so of course we want to use the alternating series bound. Let $E_N(x)$ be the error in approximating x when only the first N terms are added up. By the alternating series bound, we have

$$E_N(x) \leqslant |a_{N+1}|,$$

where the right hand side is the absolute value of the coefficient of the $(N+1)$-st power of x in the series. Clearly, we have

$$|a_{N+1}| = \frac{1}{3(N+1)^2 - 1},$$

so we want to solve for N in the right side to be smaller than 0.01. This gives

$$\frac{1}{3(N+1)^2 - 1} < 0.01,$$

so $3(N+1)^2 - 1 > 100$, meaning $3(N+1)^2 > 101$, which gives $N > \sqrt{\frac{101}{3}} - 1$.

Problem 4

(**10** Points) Five series are given below. Write down which of them converge absolutely, converge conditionally, or diverge. You don't have to show much work here, just a brief line (eg: diverges by limit comparison to $\sum \frac{1}{n}$, or diverges by ratio test) will suffice. Each answer is worth two points, but there is **no partial credit** for incorrect responses.

Part A. $\sum_{n=1}^{\infty} \frac{n - \ln(n)}{\sqrt[3]{n^2 + n - 7\ln(n+5)}}$

Diverges by n-th term test: as n is made large, the numerator behaves like n and the denominator like $n^{\frac{2}{3}}$, so overall the terms behave like $n^{1/3}$ which certainly does not go to zero for large n.

Part B. $\sum_{n=1}^{\infty} \left(\frac{n^2-1}{n^2+3}\right)^n$

Root test doesn't work (it gives $\rho = 1$), but this also diverges by the n-th term test, since the sequence of terms will converge to e^{stuff} rather than zero.

Part C. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}}{\sqrt[3]{n^2-5}}$

This is an alternating series, so we only have to check that the (absolute values of the) terms are going to zero as $n \to \infty$. But this is clearly true, just compare with leading terms $\sum \frac{n^{1/2}}{n^{2/3}} = \sum n^{-1/6}$. Thus, the series converges. On the other hand, the sum of absolute values does not converge by the p-test (where $p = \frac{1}{6} < 1$), so the convergence is conditional.

Part D. $\sum_{n=1}^{\infty} \frac{3^n}{5^n - n^3}$

Converges absolutely by comparison to $\sum \left(\frac{3}{5}\right)^n$. Note that this latter series is geometric, and $|3/5| < 1$.

Part E. $\sum_{n=1}^{\infty} \frac{\cos^3(e^n - 28n^2)}{n^2+2n}$

Converges absolutely by limit comparison to $\sum \frac{1}{n^2}$: the numerator is a cosine which is always smaller than 1, while the denominator behaves like n^2 for large n.