MIDTERM EXAM 1

MATH 312, SECTION 001

Problem 1

[**25** points] The LU decomposition of a matrix A is given as follows:

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \text{ and } \mathsf{U} = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Part a. [3 points] What is A?
Answer. A is just the matrix product of L and U, so

$$\begin{aligned} \mathsf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & 3 \\ -2 & -6 & 0 \\ -3 & -10 & 0 \end{bmatrix}. \end{aligned}$$

Part b. [3 points] Write down the sequence of row operations which takes A to U when performing Gaussian elimination.

Answer. Since $\mathsf{A} = \mathsf{LU}$, we must have $\mathsf{L}^{-1}\mathsf{A} = \mathsf{U}$ so the matrix which accomplishes our row operations is L^{-1}. One could compute this inverse via Gauss-Jordan elimination and so forth, but there is an easier way: we can just look at what L does to U and reverse that. Looking at the entries below the diagonal in L, we see L performs the following three operations:

$$\mathsf{R}_2' = \mathsf{R}_2 - 2\mathsf{R}_1, \quad \mathsf{R}_3' = \mathsf{R}_3 - 3\mathsf{R}_1 \text{ and } \mathsf{R}_3'' = \mathsf{R}_3' + \mathsf{R}_2'.$$

So, the sequence of row operations which takes us from A to U will be:

$$\mathsf{R}_2' = \mathsf{R}_2 + 2\mathsf{R}_1, \quad \mathsf{R}_3' = \mathsf{R}_3 + 3\mathsf{R}_1 \text{ and } \mathsf{R}_3'' = \mathsf{R}_3' - \mathsf{R}_2'.$$

That's it, we're done!

Part c. [10 points] Describe how you would solve $\mathsf{A}x = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ as two triangular systems[1]. Can we solve these two triangular systems in any order, or must one of them be solved before the other? Explain your answer.

Answer. Well, $\mathsf{A}x = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ means $\mathsf{LU}x = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$. So if we set $\mathsf{U}x = y$, then we are solving the two triangular systems

$$\mathsf{L}y = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right] \text{ and } \mathsf{U}x = y.$$

Of course, we must solve the L-system first to get y, otherwise there will be no right hand side to solve the U system!

[1] You don't actually have to solve anything: just explain how you'd set up the two triangular systems

Part d. [5 points] Compute the inverse of U using Gauss-Jordan elimination.

We have

$$[\mathsf{U} \mid \mathrm{Id}] = \begin{bmatrix} 1 & 4 & 3 & \mid & 1 & 0 & 0 \\ 0 & 2 & 6 & \mid & 0 & 1 & 0 \\ 0 & 0 & 3 & \mid & 0 & 0 & 1 \end{bmatrix}.$$

Perform the following operations: $\mathsf{R}_1' = \mathsf{R}_1 - 2\mathsf{R}_2$ and then $\mathsf{R}_2' = \mathsf{R}_2 - 2\mathsf{R}_3$ to get

$$\begin{bmatrix} 1 & 0 & -9 & \mid & 1 & -2 & 0 \\ 0 & 2 & 0 & \mid & 0 & 1 & -2 \\ 0 & 0 & 3 & \mid & 0 & 0 & 1 \end{bmatrix}.$$

Finally, $\mathsf{R}_1' = \mathsf{R}_1 + 3\mathsf{R}_3$ gives us

$$\begin{bmatrix} 1 & 0 & 0 & \mid & 1 & -2 & 3 \\ 0 & 2 & 0 & \mid & 0 & 1 & -2 \\ 0 & 0 & 3 & \mid & 0 & 0 & 1 \end{bmatrix},$$

and all that remains to do is scale R_2 by $1/2$ and R_3 by $1/3$. Now U^{-1} is the right hand side of the following.

$$\begin{bmatrix} 1 & 0 & 0 & \mid & 1 & -2 & 3 \\ 0 & 1 & 0 & \mid & 0 & 1/2 & -1 \\ 0 & 0 & 1 & \mid & 0 & 0 & 1/3 \end{bmatrix}.$$

Part e. [4 points] Compute A^{-1}, or explain why A is not invertible.

Answer. Note that L is always invertible, being a product of elementary matrices; and from **Part d** we know that U is invertible as well. Since $\mathsf{A} = \mathsf{LU}$, not only must A be invertible, but we must also have $\mathsf{A}^{-1} = \mathsf{U}^{-1}\mathsf{L}^{-1}$. We already have U^{-1} from **Part d**, so it remains to compute L^{-1}. We multiply the matrices corresponding to the row operations in **Part b**:

$$\mathsf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

So, we now have

$$\mathsf{A}^{-1} = \mathsf{U}^{-1}\mathsf{L}^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \\ 0 & -3/2 & -1 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}.$$

Problem 2

[**40** Points] The matrix B equals MR, where

$$\mathsf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix} \text{ and } \mathsf{R} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

You may also use the fact that

$$\mathsf{M}^{-1} = \begin{bmatrix} 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}$$

Part a. [5 points] Find a basis for the row space $C(B^T)$.

The pivoted rows of R provide a basis for the row space of B, so a basis for $C(B^T)$ is $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Part b. [10 points] Find a basis for the column space $C(B)$.

Answer. A basis for the column space of R is given by the pivot columns $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; the matrix M takes this to a basis for the column space of B. But the multiplication of M with these vectors only extracts the corresponding columns from M, so one basis for $C(B)$ is $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

Part c. [10 points] Find a basis for the null space $N(B)$.

Answer. If we label the variables of $\mathbb{R}^4$ by w, x, y and z then we see in R that the columns corresponding to y and z are free whereas those corresponding to w and x are not. Expressing w and x in terms of the free variables (using the first two – pivoted – rows of R) gives

$$w = -3y - 2z \text{ and } x = -y.$$

Therefore, a basis for $N(B)$ is given by $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Part d. [10 points] Find a basis for the left null space $N(B^T)$. **Hint**: you might need M^{-1} for this part.

A basis for $N(B^T)$ is given by extracting those rows of M^{-1} which correspond to the zero rows of R. Since only the third row of R is zero, our basis is given by the third row of M^{-1}, i.e., $\begin{bmatrix} -1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$.

Part e. [5 points] State the fundamental theorem of linear algebra, and show that B satisfies it.

The FTLA states the following: let A be any $m \times n$ matrix whose *rank* (or $\dim C(A)$) equals r. Then, $\dim N(A) = n - r$, $\dim C(A^T) = r$ and $\dim N(A^T) = m - r$. In our case, B is a 3×4 matrix with rank 2 (from **Part b**), and the corresponding dimensions are

- $\dim N(B) = 4 - 2 = 2$ from **Part c**,
- $\dim C(B^T) = 2$ from **Part a**, and
- $\dim N(B^T) = 3 - 2 = 1$ from **Part d**.

Thus, B satisfies the FTLA.

Problem 3

[**14** Points] Let $B = MR$ be the matrix from Problem 2.

Part a. [7 points] Find *all* solutions to $Bx = v$ when $v = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$. **Hint:** you *don't* have to compute the RREF of $\begin{bmatrix} B \mid v \end{bmatrix}$ from scratch: you can just use $\begin{bmatrix} R \mid M^{-1}v \end{bmatrix}$.

Answer. The vector $M^{-1}v$ equals $1/4 \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The augmented RREF $[B \mid v]$ is therefore $[R \mid M^{-1}v]$, or

$$\begin{bmatrix} 1 & 0 & 3 & 2 & \mid & 0 \\ 0 & 1 & 1 & 0 & \mid & 1 \\ 0 & 0 & 0 & 0 & \mid & -1 \end{bmatrix}.$$

It is clear from the last row that this system has **no solutions**: no linear combination of zeros can produce that minus one.

Part b. [7 points] Find *all* solutions to $Bx = v$ when $v = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$. **Hint:** see the hint given for Part a.

Answer. This time the vector $M^{-1}v$ equals $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which is much more promising. The augmented RREF $[B \mid v]$ is $[R \mid M^{-1}v]$, or

$$\begin{bmatrix} 1 & 0 & 3 & 2 & \mid & 1 \\ 0 & 1 & 1 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \end{bmatrix}.$$

Labeling the variables as w, x, y and z as in **Part b** of Problem 2, we have to satisfy the following two equations which express the pivot variables in terms of the free ones:

$$w = 1 - 3y - 2z \text{ and } x = -y.$$

So, the general solution is given by all choices of y and z in the following sum:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} z.$$

As you might expect, the last two terms are exactly the null space of B!

Problem 4

[**21** points, 3 points each] In each of the following cases, clearly mark the statement as **true** or **false**. Please also *explain* your answers in order to receive credit for this problem!

a. If a 3×4 matrix has a RREF with only three pivots, then its rows are linearly dependent.

False. There are three rows with three pivots, so they can't be dependent.

b. If a 3×4 matrix has a RREF with three pivots, then its columns must span $\mathbb{R}^3$.

True. The pivot columns must be exactly the standard basis vectors for $\mathbb{R}^3$.

c. If a matrix A satisfies $Ax = 0$ for some $x \neq 0$ then A cannot be invertible.

True. If A were invertible, then A^{-1} would send 0 to $x \neq 0$.

d. The set A consisting of the X axis, the Y axis, the line $y = x$ and the line $y = -x$ forms a subspace of $\mathbb{R}^2$.

False. This set contains $(1, 0)$ and $(0, 2)$ but not the sum $(1, 2)$.

e. If a vector k lies in the null space of A^T and if $Ax = b$ then $A(x + k)$ also equals b.

False. This would be true if k was in the null space of A, not A^T. If A is not a square matrix, then we may not even be able to add x and k because the dimensions won't match up.

f. The product of three invertible 3×3 matrices is always invertible.

True. If $M = ABC$ and all the matrices on the right side are invertible, then $M^{-1} = C^{-1}B^{-1}A^{-1}$.

g. If E is the elementary matrix which adds 3 times Row 1 to Row 2, then E^2 adds 9 times Row 1 to Row 2.

False. The matrix E^2 just performs this operation twice, with the end result of adding 6 (not 9) times Row 1 to Row 2.