MIDTERM EXAM 2

MATH 312, SECTION 001

Name:

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will *not* receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8×11 cheat-sheet.

Problem Number	Possible Points	Points Earned
1	25	
2	40	
3	20	
4	15	
Total	100	

Warning: Show your work. Answers given without proper justification will not receive full credit. In fact, they may receive no credit at all. Please explain clearly how you obtained your answers.

Problem 1

[**25** points] Consider the linear differential system

$$x' = x + 3y$$
$$y' = 2x + 2y.$$

Part a. [4 points] For which matrix A can we rewrite this system as $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]' = A \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$?

Part b. [9 points] Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

Part c. [4 points] Write the matrix exponential e^{At} as a single matrix.

Part d. [8 points] Find the solutions $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = -5$ and $y(0) = 5$.

Problem 2

[**40** Points] The matrix A is given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Part a. [6 points] Find the eigenvalues and each corresponding **unit** eigenvector for $A^T A$.

Part b. [3 points] What are the eigenvalues of AA^T?

Part c. [8 points] Find **unit** eigenvectors of AA^T corresponding to each eigenvalue found in **Part b** above.

Part d. [15 points] Find orthogonal matrices U, V and a diagonal matrix D so that $\mathsf{A} = \mathsf{U}\mathsf{D}\mathsf{V}^{\mathsf{T}}$ is the **singular value decomposition** of A. Please explain clearly how you obtain these matrices.

Part e. [8 points] Use the SVD from **Part d** to find orthonormal bases for the null space $\mathsf{N}(\mathsf{A})$, the left nullspace $\mathsf{N}(\mathsf{A}^{\mathsf{T}})$, the column space $\mathsf{C}(\mathsf{A})$ and the row space $\mathsf{C}(\mathsf{A}^{\mathsf{T}})$ of A. Clearly describe which parts of the SVD matrices you are using to extract which basis.

Problem 3

[**20** Points] A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Part a. [5 points] Apply the **Gram-Schmidt process** to find two **orthonormal** vectors u_1 and u_2 which also span V.

Part b. [5 points] Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

Part c. [10 points] Find the best possible (i.e., least squared error) solution to the linear system

$$Q \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 4

[**15 points**] Decide whether each of the following five statements is **true** or **false**. In order to receive full credit, you must provide clear and correct justification for your answers.

Part a. [3 points] If A is a 3×3 matrix with determinant 1, then $2A$ has determinant 6.

Part b. [3 points] If v and w are eigenvectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ corresponding to distinct eigenvalues, then $v^T w = 0$.

Part c. [3 points] If A is a square matrix, and if we obtain B from A via the row operation $R_2' = R_2 + 3R_1$ then B has exactly the same eigenvalues as A.

Part d. [3 points] If $A^2 = 0$ for some square matrix A then all eigenvalues of A must be zero.

Part e. [3 points] If $\det(A) = -1$ for some square matrix A, then there is some b for which $Ax = b$ has infinitely many solutions.

For Scratchwork