MIDTERM EXAM 2 SOLUTIONS

Problem 1

[**25** points] Consider the linear differential system

$$x' = x + 3y$$
$$y' = 2x + 2y.$$

Part a. [4 points] For which matrix A can we rewrite this system as $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]' = A \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$?

Ans. Clearly, if $A = \left[\begin{smallmatrix} 1 & 3 \\ 2 & 2 \end{smallmatrix}\right]$ then $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]' = A \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Part b. [9 points] Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

Ans. This is a diagonalization problem, so we compute eigenvalues and eigenvectors: solving $\det(A - \lambda I) = 0$ gives the polynomial

$$(1 - \lambda)(2 - \lambda) - 6 = 0,$$

which reduces to $\lambda^2 - 3\lambda - 4 = 0$ and hence $(\lambda - 4)(\lambda + 1) = 0$. So the two eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$. An eigenvector v_1 for $\lambda_1 = 4$ is just something nonzero in the null space of $(A - 4I)$, i.e., of $\left[\begin{smallmatrix} -3 & 3 \\ 2 & -2 \end{smallmatrix}\right]$; for instance $v_1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ will do the job. Similarly, an eigenvector v_2 for $\lambda_2 = -1$ must be chosen from the null space of $\left[\begin{smallmatrix} 2 & 3 \\ 2 & 3 \end{smallmatrix}\right]$. A fine choice would be $v_2 = \left[\begin{smallmatrix} -3 \\ 2 \end{smallmatrix}\right]$. Putting all of this together, we get

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } S = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}.$$

Part c. [4 points] Write the matrix exponential e^{At} as a single matrix.

Ans. We will use the formula that comes from diagonalization: $e^{At} = Se^{Dt}S^{-1}$. We need to compute S^{-1} first, but this is easy because S is only 2×2:

$$S^{-1} = \frac{1}{2 - (-3)} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}.$$

Now, we have a product of three matrices (and a scalar that can be pulled out):

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2e^{4t} + 3e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{4t} - 2e^{-t} & 3e^{4t} + 2e^{-t} \end{bmatrix}.$$

Part d. [8 points] Find the solutions $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = -5$ and $y(0) = 5$.

Ans. The key is to realize $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$. The exponential e^{At} has been computed in the previous part while the values of $x(0)$ and $y(0)$ are given in the question – in fact, they conveniently help us to get rid of that annoying $\frac{1}{5}$ scalar. So,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2e^{4t} + 3e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{4t} - 2e^{-t} & 3e^{4t} + 2e^{-t} \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} e^{4t} - 6e^{-t} \\ e^{4t} + 4e^{-t} \end{bmatrix}.$$

Problem 2

[**40** Points] The matrix A is given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Part a. [6 points] Find the eigenvalues and each corresponding **unit** eigenvector for A^TA.

Ans. We have

$$A^TA = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},$$

so its eigenvalues are given by solutions to $(2-\lambda)^2 - 1 = 0$, or $\lambda^2 - 3\lambda + 3 = 0$. Factoring this polynomial leads to eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. Now, a unit eigenvector v_1 for $\lambda_1 = 3$ comes from the null space of $[\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}]$, so let's choose $1/\sqrt{2}\,[\begin{smallmatrix}1\\1\end{smallmatrix}]$. Similarly, a unit eigenvector v_2 for $\lambda_2 = 1$ is just $v_2 = 1/\sqrt{2}\,[\begin{smallmatrix}1\\-1\end{smallmatrix}]$ chosen from the null space of $[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}]$. So,

$$\lambda_1 = 3 \text{ has unit eigenvector } v_1 = 1/\sqrt{2}\,[\begin{smallmatrix}1\\1\end{smallmatrix}] \text{ and}$$
$$\lambda_2 = 1 \text{ has unit eigenvector } v_2 = 1/\sqrt{2}\,[\begin{smallmatrix}1\\-1\end{smallmatrix}].$$

Part b. [3 points] What are the eigenvalues of AA^T?

Ans. The nonzero eigenvalues of AA^T must coincide with those of A^TA; but since AA^T is 3×3 rather than 2×2, it has one extra eigenvalue. Therefore the eigenvalues of AA^T must be $\lambda_1 = 3, \lambda_2 = 1$ and $\lambda_3 = 0$

Part c. [8 points] Find **unit** eigenvectors of AA^T corresponding to each eigenvalue found in **Part b** above.

Ans. Note that

$$A^TA = \begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 2\end{bmatrix},$$

and we want to find three eigenvectors u_1, u_2 and u_3 chosen from $N(A^TA - \lambda I)$ for $\lambda = 3, 1$ and 0 respectively. Let's find the eigenvector for $\lambda = 3$ here – the others can be found in a similar manner. So, we examine the matrix $A^TA - 3I$ which looks like

$$\begin{bmatrix}-2 & 0 & 1\\ 0 & -2 & 1\\ 1 & 1 & -1\end{bmatrix}.$$

We perform the following row operations: interchange rows 1 and 3, then add twice row 1 to row 3, then add row 2 to row 3. This looks like:

$$\begin{bmatrix}1 & 1 & -1\\ 0 & -2 & 1\\ -2 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1\\ 0 & -2 & 1\\ 0 & 2 & -1\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1\\ 0 & -2 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

We could work further and get this thing down to reduced row echelon form, but that won't be necessary – a triangular system suffices if all we want is a basis vector for the null space. For instance, $\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right]$ will do nicely. To make this a unit vector, let's divide by the length to get the first eigenvector: $u_1 = 1/\sqrt{6}\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right]$.

Performing similar calculations with $\lambda_2 = 1$ and $\lambda_3 = 0$ gives us the other two eigenvectors as well:

$$\lambda_1 = 3 \text{ has unit eigenvector } u_1 = 1/\sqrt{6}\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right],$$
$$\lambda_2 = 1 \text{ has unit eigenvector } u_2 = 1/\sqrt{2}\left[\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right], \text{ and}$$
$$\lambda_3 = 0 \text{ has unit eigenvector } u_3 = 1/\sqrt{3}\left[\begin{smallmatrix}1\\1\\-1\end{smallmatrix}\right].$$

Part d. [15 points] Find orthogonal matrices U, V and a diagonal matrix D so that $A = UDV^T$ is the **singular value decomposition** of A. Please explain clearly how you obtain these matrices.

Ans. The SVD of A is given by $A = UDV^T$, where

$$D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has the same shape as A but contains square-roots of the common eigenvalues – that is, $\lambda_1 = 3$ and $\lambda_2 = 1$ – of AA^T and A^TA in descending order along its main diagonal. Now, U contains the unit eigenvectors of AA^T in the same order as the eigenvalues – so, u_1, u_2 and u_3 become the three columns of U:

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{bmatrix}$$

And finally, V inherits its columns from the eigenvectors of AA^T again in the same order v_1, v_2:

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Part e. [8 points] Use the SVD from **Part d** to find orthonormal bases for the null space $N(A)$, the left nullspace $N(A^T)$, the column space $C(A)$ and the row space $C(A^T)$ of A. Clearly describe which parts of the SVD matrices you are using to extract which basis.

Ans. A basis for $N(A)$ would be given by columns of V associated to the zero eigenvalues – but of course, there are no such columns. The null space is therefore trivial (i.e., it is the zero vector space) and has no basis whatsoever. The two columns of V in fact produce a basis $\{1/\sqrt{2}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], v_1 = 1/\sqrt{2}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\}$ for $C(A^T)$, which must equal all of $\mathbb{R}^2$.

The matrix U, on the other hand, does have its last column corresponding to the zero eigenvalue λ_3 of AA^T, so this column $1/\sqrt{3}\left[\begin{smallmatrix}1\\1\\-1\end{smallmatrix}\right]$ is a basis of $N(A^T)$. The first two columns $\left\{1/\sqrt{6}\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right], 1/\sqrt{2}\left[\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right]\right\}$ of U correspond to nonzero eigenvalues and hence form a basis of $C(A)$.

Problem 3

[**20** Points] A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Part a. [5 points] Apply the **Gram-Schmidt process** to find two **orthonormal** vectors u_1 and u_2 which also span V.

Let u_1 and u_2 be the two columns of A. We apply Gram-Schmidt in two stages: first, we only produce orthogonal vectors w_1 and w_2 which span V, not caring about their lengths. Next, we will divide w_1 and w_2 by their respective lengths to get u_1 and u_2. With this in mind, note that the first step of Gram-Schmidt is easy:

$$w_1 = v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

no work needed. The next step involves finding w_2, which is a bit harder. Recall that

$$w_2 = v_2 - \mathrm{Proj}_{v_1} v_2,$$

so we must subtract from v_2 its orthogonal projection onto v_1. But this projection is given by

$$\mathrm{Proj}_{v_1} v_2 = \frac{v_1^T v_2}{v_1^T v_1} v_1 = \frac{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

So, we have

$$w_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

Now all we have to do is divide w_1 and w_2 by their lengths, so the desired orthonormal vectors are:

$$u_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ and } u_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Part b. [5 points] Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

Ans. We just want $Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ where u_1 and u_2 are the orthonormal vectors from the previous answer. Since V is the column space $C(A)$ which equals $C(Q)$ by the basic property of Gram-Schmidt, the matrix which projects onto V is given by

$$P_V = Q(Q^T Q)^{-1} Q^T,$$

but by orthogonality of Q that $(Q^T Q)^{-1}$ bit in the middle is just the identity, so the projection matrix becomes QQ^T.

Part c. [10 points] Find the best possible (i.e., least squared error) solution to the linear system

$$Q\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Ans. The least-squared error would be given by solutions to the normal equations

$$Q^T Q \begin{bmatrix} x \\ y \end{bmatrix} = Q^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

but since $Q^T Q$ is just the identity, our solution is just $Q^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, or

$$\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 5/\sqrt{6} \end{bmatrix}.$$

Problem 4

[15 points] Decide whether each of the following five statements is **true** or **false**. In order to receive full credit, you must provide clear and correct justification for your answers.

Part a. [3 points] If A is a 3×3 matrix with determinant 1, then $2A$ has determinant 6.

Ans. This is **false**. Scaling A by 2 scales each of the three rows of A by 2, which scales the determinant by $2^3 = 8$, not 6.

Part b. [3 points] If v and w are eigenvectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ corresponding to distinct eigenvalues, then $v^T w = 0$.

Ans. This is **true** by the spectral theorem: our matrix A is symmetric.

Part c. [3 points] If A is a square matrix, and if we obtain B from A via the row operation $R_2' = R_2 + 3R_1$ then B has exactly the same eigenvalues as A.

Ans. This is **false**: just look at $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Part d. [3 points] If $A^2 = 0$ for some square matrix A then all eigenvalues of A must be zero. **Hint:** Start with $Av = \lambda v$.

Ans. This is **true**. If λ is an eigenvalue of A, then $Av = \lambda v$ for some nonzero vector v. But then, $A^2 v = A(\lambda v) = \lambda(Av) = \lambda^2 v$. Since $\lambda^2 v = 0$ for some nonzero v, we must have $\lambda = 0$.

Part e. [3 points] If $\det(A) = -1$ for some square matrix A, then there is some b for which $Ax = b$ has infinitely many solutions.

Ans. This is **false**. Since the determinant is nonzero, A is invertible. So, $x = A^{-1}b$ is the unique solution to $Ax = b$.