# HOMEWORK ASSIGNMENT 8

**Name**: **Due:** Never!

## Problem 1

Find the maximum value taken by $f(x, y) = 2x + 3y$ for positive $x$ and $y$ subject to the three constraints $x \leqslant 4, x + y \leqslant 10$ and $y \leqslant 6$ using only geometric techniques. More precisely,

(1) Draw the feasible region $D$ as a subset of $\mathbb{R}^2$
(2) Identify the five corner points of $D$: what are the $(x, y)$ values of these corner points?
(3) Evaluate $f$ at each of the corner points and find the maximum.

## Problem 2

Now we will solve the optimization problem from **Problem 1** using the *Simplex method.*

(1) Express the optimization problem in standard form: meaning, find $A$, $b$ and $c$ so that we are being asked to maximize $c^T \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ subject to $A \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \leqslant b$ and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \geqslant 0$.
(2) Write down the augmented block matrix

$$B = \begin{bmatrix} 1 & -c^T & 0 & \vdots & 0 \\ 0 & A & \mathrm{Id} & \vdots & b \end{bmatrix}.$$

The penultimate three columns correspond to slack variables $r, s, t \geqslant 0$. At this initial stage, these three are pivot variables while $x$ and $y$ are not.

(3) On $B$, perform the row operations needed by the simplex algorithm of $B$. Carefully explain how you are selecting each column and row to produce a new pivot.

## Problem 3

Throughout this problem, assume that $N = 6$.

(1) What is the primitive $N$-th root $\omega_N$ of 1?
(2) Write down the discrete Fourier matrix $D_N$ in terms of negative powers of $\omega_N$. Simplify so that no power is smaller than $-5$. (So it is okay to populate the matrix with entries like $\omega_6^{-2}$ but not something like $\omega_6^{-9}$ and *definitely* not something awful like $\left(\frac{1-\sqrt{3}i}{2}\right)^{-2}$.
(3) Write down the $6 \times 6$ matrices $A_N$ and $B_N$ so that

$$D_N = A_N \begin{bmatrix} D_{N/2} & 0 \\ 0 & D_{N/2} \end{bmatrix} B_N,$$

which come from the Cooley-Tukey fast Fourier transform algorithm.

## Problem 4

In this problem, we have $N = 4$.

(1) Write down the discrete Fourier matrix $D_4$ and explain how you have obtained its entries.
(2) There are only two distinct columns of $D_4$ which *don't* contain exclusively real numbers. Identify these two columns, and check that they are orthogonal.
(3) What is the inverse matrix $D_4^{-1}$?

## Problem 5

Given the matrix

$$A = \begin{bmatrix} 5 & -2 & -3 \\ -1 & 4 & -3 \\ 1 & -4 & 3 \end{bmatrix},$$

(1) Compute all the eigenvalues of $A$ and write down their algebraic multiplicities.
(2) Compute eigenvectors corresponding to the eigenvalues of $A$: what are the geometric multiplicities of the eigenvalues?
(3) Is $A$ diagonalizable? If yes, write it as $SDS^{-1}$. Otherwise, explain why we can't find $S$ and $D$.
(4) Compute the Jordan decomposition of $A$ – that is, find matrices $S$ and $J$ so that $S$ is invertible, $J$ is in Jordan form, and $A = SJS^{-1}$.

## Problem 6

Consider a $k \times k$ Jordan block

$$M = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

Show that if $M^2 = M$ then $k$ must equal 1, and $\lambda$ must be either 0 or 1. (**Hint:** Assume $k = 2$ and compute $M^2$ for an arbitrary $\lambda$, and set it equal to $M$. The argument for general $k$ is quite similar!).