Extra Problems (Subspaces)

1. Show that all Polynomials form a vector space $\underline{P}$

2. Show that all polynomials with degree no more than 3 form a vector subspace in $\underline{P}$.

3. Show that all polynomials that takes the value 0 at the point 0 form a vector subspace in $\underline{P}$.

4. Consider the set $S = \{\text{polynomials with degree } 2\} \cup \{0\}$, show that even $0 \in S$, S is still NOT a subspace.

5. Consider the following subsets of $\mathbb{R}^3$, which of them form subspaces?

   a) $\{(x, y, z) \mid x + y + z = 0\}$

   b) $\{(x, y, z) \mid x + y + z^2 = 0\}$

   c) $\{(x, y, z) \mid x, y, z \text{ are rational numbers}\}$.

   d) $\{(x, y, z) \mid xyz = 0\}$.

   e) $\{(x, y, z) \mid x = 3t,\ y = 2t,\ z = t \text{ for } t \in \mathbb{R}\}$.

1. 2. 3. are proof-related, which is not required, still, you may try to do them, only #2 is <u>non-trivial</u>.

4. consider two elements in S:

$p_1(x) = x^2 + 1$

$p_2(x) = -x^2 + 2x$,

both of them have degree exactly 2.

but $p_1(x) + p_2(x) = 2x+1$ has degree only 1. so does not lie in S.

5. a) this is subspace. it's a <u>plane</u>.

b) <u>No</u>. consider $p = (-2, -2, 2)$ an element. multiply by a scalar -1, we get $q = (2, 2, -2)$. but

$2 + 2 + (-2)^2 = 8 \neq 0$, so $q = -p \notin S$.

c) <u>No</u>. choose $(1.1.1) \in S$, multiply by a scalar π which is <u>irrational</u>, we get $(\pi, \pi, \pi) \notin S$.

d) <u>No</u>. the set is the <u>union</u>: $\{x=0\} \cup \{y=0\} \cup \{z=0\} = S$.

consider two points $(1, 0, 6)$, $(0, 1, 0)$

their sum is $(1, 1, 6)$ which is not in S

e) this is a subspace. it's a line parametrized by z.

in another form. $S = \text{Span}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, so indeed it's a subspace spanned by a single element $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Extra Problems (matrix reduction, linear Equation)

1. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. find the RREF and keep track of the row operations.

$\boxed{\text{Solution}}$:

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_3'' = R_3' - \frac{1}{2}R_2'} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

a) compute $N(A)$, $C(A)$.

$\boxed{\text{Solution}}$ note $N(A) = N(U)$ with $U = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

but U is full rank, so $N(U) = \{0\}$ therefore $N(A) = \{0\}$.

though in general $C(A)$ is not the same as $C(U)$.

but when U is full rank, A is full rank,

So the columns are all independence. therefore

$C(A) = \mathbb{R}^3$.

b) when $b = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$, find the solution of the equation

$\boxed{\text{Solution}}$ we compute $E_2 E_1 b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -8 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 2 \end{pmatrix}.$$ So we get:

$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 2 \end{pmatrix}$

we get $\begin{cases} x - y + z = 4 \\ 4y - 2z = -8 \\ z = 2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$ as the soluty.

2. given $A = \begin{pmatrix} 3 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ compute the following:

$N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$

$\boxed{\text{Solution}}$: Note A is already in RREF.

1) $N(A)$. note $\underline{z}$ is a free-variable.

$$\begin{cases} 3x + z + 2w = 0 \\ 2y + z + 3w = 0 \\ -w = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{3}z \\ y = -\frac{1}{2}z \\ z = z \\ w = 0 \end{cases}$$ set $z=1$, we get

$$N(A) = \text{Span}\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}.$$

2) $C(A)$ is spanned by pivot columns. so

$$C(A) = \text{Span}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.$$

3). write $A^T = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix}$, again $C(A^T)$ is spanned by pivot columns, (though we think of pivot as "0" above, not below).

so $C(A^T) = \text{Span}\begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

4) Let's calculate the RREF for A^T:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xRightarrow[\text{z is free variable.}]{} \begin{pmatrix} 0 \\ 0 \\ z \\ 0 \end{pmatrix}$$, so set $z=1$, we get.

$$N(A^T) = \text{Span}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$