1. $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$ $\begin{matrix} R_2 - 4R_1 \\ R_3 + 2R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$ $R_3 - 2R_2$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

↑ upper triangular matrix.

$E_{32} E_{31} E_{21} A = U.$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1' = R_1 + R_3$ (add row 3 to row 1).

10

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $R_1' = R_1 + R_3$ (and) $R_3' = R_1 + R_3.$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

2. Additional question: matrix in (b) is not invertible because R_1 and R_3 are identical, which reduces the matrix into 2×3 matrix, which is not invertible.

3. $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ $\qquad$ $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3b \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix}$$

predictions:

$$A^5 = \begin{bmatrix} 1 & 5b \\ 0 & 1 \end{bmatrix} \qquad A^5 = \begin{bmatrix} 32 & 32 \\ 0 & 0 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix} \qquad A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$$

4. $Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ $Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad$ $Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[x_1\ x_2\ x_3] = X$ $\qquad$ $AX = A[x_1\ x_2\ x_3]$

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{1}{2}R_1 \end{matrix} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$R_3 - \frac{1}{3}R_2 \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \frac{3}{4}R_3 \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$\begin{matrix} R_2 - \frac{1}{2}R_3 \\ R_1 - R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \frac{2}{3}R_2 \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$R_1 - R_2 \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \frac{1}{2}R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

10

$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix} \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 2 & 0 \\ 0 & -3 & 3 & 1 & 0 & 2 \end{array}\right]$$

$$-R_3 \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 2 & 0 \\ 0 & 3 & -3 & -1 & 0 & -2 \end{array}\right] R_3 - R_2 \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{array}\right]$$

B is not invertible

7. E_{21}, E_{31}, E_{32} turn A into upper triangular matrix U.

Multiply by $E_{32}^{-1}, E_{31}^{-1}, E_{21}^{-1}$ to factor A into $LU = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$(E_{32}E_{31}E_{21})A = U. \rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U. = LU$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U.$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{31}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$= A.$$

8. LU decomposition of A (unknown) is $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$U = L^T$

Solve $Ax=b$ for $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ as 2 triangular systems.

Then, compute the original matrix A.

$$Ax = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$LU\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \rightarrow U\begin{bmatrix} a \\ b \\ c \end{bmatrix} = L^{-1}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad U = L^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} a+b+c \\ b+c \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \rightarrow c=1,\ b=0,\ a=3$$

$$x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

10

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$