HOMEWORK ASSIGNMENT 3

Name: **Due:** Monday Feb 24

PROBLEM 1: STRANG 3.1 #9, #10 PAGE 128

This problems tests your understanding of vector spaces and subspaces.

(1) Find a set of vectors in $\mathbb{R}^2$ for which $x+y$ stays in the set but $\frac{1}{2}x$ may be outside for some x in the set.
(2) Find a set of vectors in $\mathbb{R}^2$ (other than two quarter-planes) for which every cx stays in the set but $x+y$ may be outside.
(3) Is the set of vectors (b_1, b_2, b_3) with $b_1 = b_2$ a subspace of $\mathbb{R}^3$? Briefly explain why or why not.
(4) Same as above, the set of vectors with $b_1 b_2 b_3 = 0$.
(5) Same as above, the set of vectors with $b_1 \leqslant b_2 \leqslant b_3$.

Ans: 1) $S = \{(x,y) \mid x, y \text{ are integers}\}$, then $x = (1,0) \in S$, but $\frac{1}{2}x \notin S$

2) $S = \{(x,y) \mid x=0 \text{ or } y=0\}$. then $x=(1,0)$, $y=(0,1) \in S$, $x+y \notin S$.

3) Yes

4) No. $x=(1,0,0)$ $y=(0,1,1) \in S$, $x+y = (1,1,1) \notin S$

5) No. $x=(1,2,3) \in S$, $-x = (-1,-2,-3) \notin S$.

Note: There can be quite difference constructions for 1) and 2)

PROBLEM 2: STRANG 3.4 #1 PAGE 163

Describe the column space and null space of A. Also compute the complete solution to $Ax = b$

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

Ans: $A \rightsquigarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so $C(A) = \text{Span} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$.

first we compute $N(A) = \left\{ (x,y,z,w) \;\middle|\; \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

note z, w are free. we get $N(A) = \text{Span} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

a special solution is $\begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. so a general solution is

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} w.$$

Problem 3: Strang 3.4 #8 page 164

Which vectors $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ are in the column space of A? Which combinations of rows of A give the zero row? Answer these questions separately for these two choices of A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

Ans:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}. \quad \text{so } C(A) = \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{so } C(A) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right).$$

Problem 4: Strang 3.4 #18 page 165

Compute the ranks of A and A^T (these might depend on q). Show your work!

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix}.$$

Ans:

$$A \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$$

so if $q=2$. $\text{rank}(A) = \text{rank}(A^T) = 2$

if $q \neq 2$. $\text{rank}(A) = \text{rank}(A^T) = 3$.

Problem 5: Strang 3.5 #2 page 178

Find the largest possible number of linearly independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Explain how you found this number.

Ans:

v_1. v_2 v_3 are linearly independent. as can be seen by:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ is already in full-rank.

now $v_4 = v_2 - v_1$. $v_5 = v_3 - v_1$, $v_6 = v_3 - v_2$.

so the maxial number of linelly independence vectors is 3.

Problem 6: Strang 3.5 #25 page 179

Decide the dependence or independence of

(1) the vectors $(1,3,2)$, $(2,1,3)$ and $(3,2,1)$,
(2) the vectors $(1,-3,2)$, $(2,1,-3)$ and $(-3,2,1)$.

Again, explain your answers.

Ans:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{pmatrix}.$$

so they are all independence.

$$\begin{pmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

in fact. $v_1 + v_2 + v_3 = 0$. so they're linearly dependent

Problem 6: Not from Strang

The vector $\mathbf{b} = (4, 20, 14)$ equals $-3\mathbf{u} + \mathbf{v} + 5\mathbf{w}$ where $\mathbf{u} = (2, 4, 1)$, $\mathbf{v} = (0, 2, 2)$ and $\mathbf{w} = (2, 6, 3)$. Can we do any better? Does $\mathbf{b}$ lie in the span of $\mathbf{u}$ and $\mathbf{v}$ alone? Explain why or why not.

Ans:

u. v are linealy independent.

but $w = u + v$.

so $b = -3u + v + 5w$

$= -3u + v + 5(u+v)$

$= 2u + 6v$.

therefore we only need to use u, v to express b.

Problem 7: Strang 3.5 #20 page 180

Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. (Hint: both problems can be expressed in terms of finding nullspaces of certain matrices).

Ans:

$$(1\ -2,\ 3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

y. z are free. we get $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ as basis.

now if further we need $z = 0$. that is

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$ we see y is free.

this leads to $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ as a basis.

Problem 8: Not in Strang

All you know about a 3×5 matrix A is that it has rank 2. Compute $\dim N(A) - \dim N(A^T) + \dim C(A) - \dim C(A^T)$. Explain how you got your answer.

Problem 8 :

A is a 3×5 matrix with rank 2.
Compute $\dim N(A) - \dim N(A^T) + \dim C(A) - \dim C(A^T)$.
explain why.

By the fundamental theorem of linear algebra.

$$\dim N(A) = 5 - r(A) = 5 - 2 = 3$$

$$\dim C(A) = r(A) = 2$$

$$\dim N(A^T) = 3 - r(A^T) = 3 - 2 = 1$$

$$\dim C(A^T) = r(A^T) = 2$$

So. our sum $= 3 - 1 + 2 - 2 = 2$.

Problem 9: Strang 3.6 #3 page 191

Find bases for each of the four fundamental subspaces associated with A.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans:

base for $C(A)$ ~~=~~ pivot column in RREF $= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$.

base for $C(A^T)$ = pivot row in RREF $= \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

For $N(A)$. note $(x, y. z, w, u)$, we have

x, z, u are free variable. so a base is by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

For $N(A^T)$, we compute $M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

the null-row in RREF correspond to the 3rd row in M^{-1}. which is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

so $N(A^T) = \text{Span} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.