HOMEWORK ASSIGNMENT 4

Name: **Due:** Wednesday Mar 19

PROBLEM 1: STRANG 4.1 #6 PAGE 203

This system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solutions.

$$x + 2y + 2z = 5$$
$$2x + 2y + 3z = 5$$
$$3x + 4y + 5z = 9.$$

(1) Find numbers y_1, y_2 and y_3 so that scaling the first equation by y_1, the second by y_2 and the third by y_3 before adding them all up leads to the contradiction $0 = 1$.
(2) Which of A's four fundamental subspaces contains the vector $y = (y_1, y_2, y_3)$?

Ans:

1) $\begin{cases} y_1 = 1. \\ y_2 = 1 \\ y_3 = -1 \end{cases}$

therefore $(1,1,-1)$ is in the null space of A^T, i.e $N(A^T)$

2) from 1). we get $(1,1,-1) \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} = (0,0,0)$

so $(1,1,-1)$ is perpendicular to the column space of A.

PROBLEM 2: STRANG 4.1 #11 PAGE 203

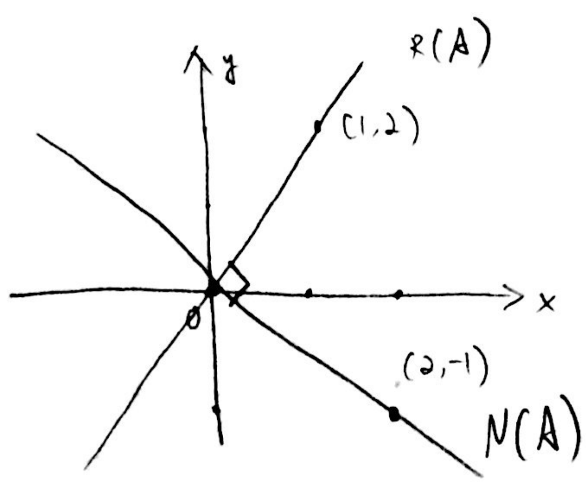
Draw and label the four fundamental subspaces for

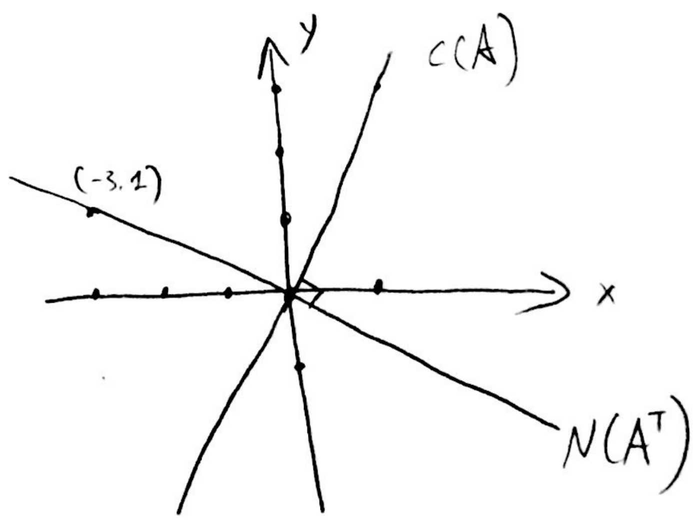
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Draw the row and null space in one figure and the column and left null space in another.

Ans: $R(A) = \text{Span}\,(1,2)^T$, the rest two fundamental spaces are determined by orthogonal relations

$C(A) = \text{Span}\begin{pmatrix}1\\3\end{pmatrix}$.





Problem 3: Strang 4.1 #22 page 204

Suppose V is spanned by the vectors (1, 2, 2, 3) and (1, 3, 3, 2). Find a basis for $V^\perp$. This is the same as solving $Ax = 0$ for which matrix A?

Let $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

$\uparrow \ \uparrow$ free.

$V^\perp = \text{Span} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

Problem 4: Strang 4.2 #10 Page 215

Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix},$$

(1) Find the matrix P which projects onto the column space of A,
(2) Compute the projection p of b onto this column space,
(3) Find the error $e = b - p$ and show that it lies in the left nullspace of A.

Ans:

1) we compute $P = A(A^TA)^{-1}A^T$

$A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$(A^TA)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

so $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2) $Pb = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

3) $e = b - p = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

$\Rightarrow e^tA = (0, 0, 4)\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, 0)$, so e is in left null sp

Problem 5: Strang 4.2 #17 page 215

If P is a square matrix with $\mathsf{P}^2 = \mathsf{P}$, show that $(\mathsf{I} - \mathsf{P})^2 = (\mathsf{I} - \mathsf{P})$ where I is the identity matrix. Hint: just multiply out $(\mathsf{I} - \mathsf{P})(\mathsf{I} - \mathsf{P})$ and use the information given already.

Ans: we compute

$$(I-P)(I-P) = I - P - P + P^2$$

$$= I - P - P + P$$

$$= I - P$$

Problem 6: Strang 4.2 #19 page 216

Choose two independent vectors lying on the plane $x - y - 2z = 0$ and make them the columns of a matrix A. Then compute the matrix $\mathsf{A}(\mathsf{A}^\mathsf{T}\mathsf{A})^{-1}\mathsf{A}^\mathsf{T}$: this matrix projects onto our plane!

Ans:

$$x = y + 2z$$

so the plane is spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

set $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$$A^TA = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \qquad (A^TA)^{-1} = \frac{1}{6}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

$$P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 2 \\ 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & \\ & & \end{pmatrix}$$

Problem 7: Strang 4.3 #6 page 227

Compute the projection of $\mathbf{b} = (0, 8, 8, 20)$ onto the line through $\mathbf{a} = (1, 1, 1, 1)$ by first finding the scalar $\mathbf{c} = \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}}$.

Ans:

$$a^T b = (1, 1, 1, 1) \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = 36$$

$$a^T a = (1, 1, 1, 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 4$$

$$c = \frac{a^T b}{a^T a} = 9$$

$$p = ca = 9a = (9, 9, 9, 9)$$

Problem 8: Strang 4.3 #9 page 227

Use the method of least squares to find the parabola $y = C + Dx + Ex^2$ which best approximates the four data points given in (x, y) format by $(0, 0)$, $(1, 8)$, $(3, 8)$ and $(4, 20)$.

Ans:

we get 4 equations: $b^T = (0, 8, 8, 20)$

$$\begin{cases} C + D \cdot 0 + E \cdot 0^2 = 0 \\ C + D \cdot 1 + E \cdot 1^2 = 8 \\ C + D \cdot 3 + E \cdot 3^2 = 8 \\ C + D \cdot 4 + E \cdot 4^2 = 20 \end{cases}$$

so $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}$$

we solve $A^T A \cdot \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$ leads to $\begin{cases} C = 2 \\ D = \frac{4}{3} \\ E = \frac{2}{3} \end{cases}$

Problem 9: Strang 4.4 #6 page 240

If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also orthogonal. Hint: use the fact that Q^TQ is the identity whenever Q is orthogonal.

Ans:

$$(Q_1Q_2)^T Q_1Q_2$$

$$= (Q_2^T Q_1^T)Q_1Q_1$$

$$= Q_2^T (Q_1^T Q_1) Q_2$$

$$= Q_2^T \cdot I \cdot Q_2$$

$$= Q_2^T Q_2$$

$$= I.$$

Problem 10: Similar to Strang 4.4 #11 page 240

Use the Gram-Schmidt method to find orthonormal vectors q_1 and q_2 in the plane spanned by $(1, 0, -1, 1, 3)$ and $(2, 3, 2, 0, 1)$.

Ans:

$$A = (1, 0, -1, 1, 3)$$

$$B = b - \frac{A^Tb}{A^TA} \cdot A$$

$$= (2, 3, 2, 0, 1) - \frac{3}{1+1+1+9} \cdot (1, 0, -1, 1, 3)^t$$

$$= (2, 3, 2, 0, 1) - \frac{1}{4}(1, 0, -1, 1, 3)$$

$$= \left(\frac{7}{4}, 3, \frac{9}{4}, -\frac{1}{4}, \frac{1}{4}\right)$$

$$q_1 = \frac{A}{\sqrt{A^TA}} = \frac{1}{2\sqrt{3}}(1, 0, -1, 1, 3)$$

$$q_2 = \frac{B}{\sqrt{B^TB}} = \frac{1}{2\sqrt{69}}(7, 12, 9, -1, 1.)$$

Problem 11: Strang 4.4 #15 page 241

Given the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix},$$

(1) Find three orthonormal vectors q_1, q_2 and q_3 so that q_1 and q_2 span the column space of A.
(2) Which of the four fundamental subspaces contains q_3?
(3) Solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ by least squares. Hint: it will *greatly* simplify computations if you use the orthonormal basis for $C(A)$!

Ans: 1) $\tilde{q}_1 = \frac{q_1}{\|q_1\|} = \frac{1}{3}\cdot\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $q_2' = q_2 - \frac{q_1^T q_2}{q_1^T q_1}\cdot q_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\tilde{q}_2 = \frac{q_2'}{\|q_2'\|} = \frac{1}{3}\cdot\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

$\tilde{q}_3 = \tilde{q}_1 \times \tilde{q}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

2) $\tilde{q}_3 \in N(A^T)$ since it satisfies $\begin{pmatrix} \tilde{q}_1^T \\ \tilde{q}_2^T \end{pmatrix} q_3 = 0$. i.e $A^T q_3 = 0$

3) $Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{pmatrix}$, then we need to solve $Q^T Q x = Q^T b$ with $b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$

$Q^T Q = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$. So. $\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9 \\ 18 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Problem 12: Strang 4.4 #31 page 243

Consider the matrix

$$Q = c\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

(1) Choose c so that Q becomes an orthogonal matrix.
(2) Project $b = (1, 1, 1, 1)$ onto the line spanned by the first column of Q.
(3) Project b onto the plane spanned by the first two columns of Q.

1) $Q^T Q = c^2 \cdot \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I \Rightarrow c^2 = \frac{1}{4} \Rightarrow c = \pm\frac{1}{2}$.

2) $b = (1,1,1,1)$, $c = \frac{v^T \cdot b}{v^T v} = \frac{-1}{1} = -1$, $p = c \cdot v = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
$v = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

3) $P_v(1,1,1,1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.