HOMEWORK ASSIGNMENT 5

Name: **Due:** Wednesday Mar 26

PROBLEM 1: STRANG 5.1 #3 AND #28 PAGE 251, 254

State true or false, giving a reason when the statement is true and a counterexample when the statement is false. All matrices involved are $n \times n$ where $n > 1$.

(1) $\det(I + A) = 1 + \det(A)$.
(2) $\det(ABC) = \det(A)\det(B)\det(C)$.
(3) $\det(4A) = 4\det(A)$.
(4) $\det(AB - BA) = 0$.
(5) If A is not invertible, AB is not invertible.
(6) $\det(A)$ always equals the product of its pivots.
(7) $\det(A - B) = \det(A) - \det(B)$.
(8) $\det(AB) = \det(BA)$.

Ans:

1) F. take $n=2$. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $\det(I+A) = \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$, $1 + \det A = 1$.

2) T. $\det(ABC) = \det(AB) \cdot \det C = \det A \cdot \det B \cdot \det C$.

3) F. $\det(4A) = 4^n \cdot \det(A)$. take $A = I$. $n > 1$

4) F. $\det\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$

5) T. $\det(AB) = \det A \cdot \det B = 0 \cdot \det B = 0$.

6) F. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ has $\det A = -1$. while the pivots multiply gives 1.

7) F. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\det A = \det B = 0$, $\det(A-B) = -1$.

8) T. $\det(AB) = \det A \cdot \det B$
$\det(BA) = \det B \cdot \det A$

Problem 2: Strang 5.1 #8 page 252

Prove that every orthogonal $n \times n$ matrix Q has determinant equal to 1 or -1. **Hint:** Use the fact that $QQ^T = I$, the product formula for determinants.

Ans: By the Hint, $QQ^T = I$

$$\det(QQ^T) = \det(I) = 1$$

$$\det(QQ^T) = \det Q \cdot \det Q^T$$
$$= \det Q \cdot \det Q$$
$$= (\det Q)^2$$

So $(\det Q)^2 = 1 \Rightarrow \det Q = \pm 1$.

Problem 3: Strang 5.1 #24 page 254

The matrix A has the following LU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}, \text{ and } U = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Doing *as little computation as possible*, find the determinants of L, U, A, U^{-1}, L^{-1} and $U^{-1}L^{-1}A$.

L, U are triangular, so the determinant is multiplying its diagonals.

$$\det L = 1 \cdot 1 \cdot 1 = 1$$

$$\det U = 3 \cdot 2 \cdot (-1) = -6.$$

$$\det A = \det L \cdot \det U = 1 \cdot (-6) = -6.$$

$$\det(U^{-1}) = (\det U)^{-1} = \frac{1}{-6} = -\frac{1}{6}.$$

$$\det(L^{-1}) = (\det L)^{-1} = \frac{1}{1} = 1.$$

$$\det(U^{-1}L^{-1}A) = \det(U^{-1}) \cdot \det(L^{-1}) \det(A)$$
$$= -\frac{1}{6} \cdot \frac{1}{1} \cdot (-6) = 1.$$

Problem 4: Strang 5.1 #27 Page 254

Given

$$\mathsf{C} = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix},$$

use row operations to compute $\det(\mathsf{C})$.

Ans:

$$C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \qquad LC = U.$$

$$\det L = 1 \cdot 1 = 1.$$

$$\det C = \det(L^{-1}) \cdot \det(U) = 1 \cdot a \cdot (b-a) \cdot (c-b)$$

$$= a(b-a)c(c-b).$$

Problem 5: Strang 5.2 #12 Page 264

Given

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

find the cofactor matrix C and compute the matrix product $\mathsf{A}\mathsf{C}^{\mathsf{T}}$. Use this product to find $\det(\mathsf{A})$.

Ans:

$$C_{11} = \det\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = 5. \qquad C_{12} = -\det\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} = 2.$$

$$C_{13} = \det\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} = 1, \qquad C_{21} = -\det\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = 2. \qquad C_{22} = \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$$

$$C_{23} = -\det\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = 2. \qquad C_{31} = \det\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = -1,$$

$$C_{32} = -\det\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = -2, \qquad C_{33} = \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3.$$

$$C = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & 3 \end{bmatrix}. \qquad AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\det A = 8.$$