Multilinear Hyperquiver Representations

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ABSTRACT. We count singular vector tuples of a system of tensors assigned to the edges of a directed hypergraph. To do so, we study the generalisation of quivers to directed hypergraphs. Assigning vector spaces to the nodes of a hypergraph and multilinear maps to its hyperedges gives a hyperquiver representation. Hyperquiver representations generalise quiver representations (where all hyperedges are edges) and tensors (where there is only one multilinear map). The singular vectors of a hyperquiver representation are a compatible assignment of vectors to the nodes. We compute the dimension and degree of the variety of singular vectors of a sufficiently generic hyperquiver representation. Our formula specialises to known results that count the singular vectors and eigenvectors of a generic tensor.

1. Introduction

The theory of quiver representations provides a unifying framework for some fundamental concepts in linear algebra [7]. In this paper, we introduce and study a natural generalisation of quiver representations, designed to analogously serve the needs of multilinear algebra.

Quiver Representations and Matrix Spectra. A *quiver* Q consists of finite sets V and E, whose elements are called vertices and edges respectively, along with two functions $s, t : E \to V$ sending each edge to its source and target vertex. It is customary to write $e : i \to j$ for the edge e with $s(e) = i$ and $t(e) = j$. The definition does not prohibit self-loops $s(e) = t(e)$ nor parallel edges $e_1, e_2 : i \to j$. A *representation* (U, α) of Q assigns a finite-dimensional vector space U_i to each $i \in V$ and a linear map $\alpha_e : U_i \to U_j$ to each $e : i \to j$ in E. Originally introduced by Gabriel to study finite-dimensional algebras [22], quiver representations have since become ubiquitous in mathematics. They appear prominently in disparate fields ranging from representation theory and algebraic geometry [12] to topological data analysis [32]. In most of these appearances, the crucial task is to classify the representations of a given quiver up to isomorphism. This amounts in practice to cataloguing the *indecomposable* representations; i.e., those that cannot be expressed as direct sums of smaller nontrivial representations.

For all but a handful of quivers, the set of indecomposables (up to isomorphism) is complicated, and such a classification is hopeless [30, Theorem 7.5]. Nevertheless, one may follow the spirit of [37] and use quivers to encode compatibility constraints with spectral interpretations. We work with representations that assign vector spaces $U_i = \mathbb{C}^{d_i}$ to each vertex and matrices $A_e : \mathbb{C}^{d_i} \to \mathbb{C}^{d_j}$ to each edge. We denote the quiver representation by $(\boldsymbol{d}, A)$, where $\boldsymbol{d} := (d_1, \ldots, d_n)$ is the dimension vector. Let $[\boldsymbol{x}] \in \mathbb{P}(\mathbb{C}^d)$ denote the projectivisation of $\boldsymbol{x} \in \mathbb{C}^d$. We define the **singular vectors** of a quiver representation $(\boldsymbol{d}, A)$ to consist of tuples $\big([\boldsymbol{x}_i] \in \mathbb{P}(\mathbb{C}^{d_i}) \mid i \in V\big)$ for which there exist scalars $(\lambda_e \mid e \in E)$ so that the compatibility constraint $A_e \boldsymbol{x}_i = \lambda_e \boldsymbol{x}_j$ holds across each edge $e : i \to j$. Standard notions from linear algebra arise as special cases of such singular vectors, see also Figure 1:

(a) The eigenvectors of a matrix $A : \mathbb{C}^d \to \mathbb{C}^d$ are the singular vectors of the representation of the Jordan quiver that assigns $\mathbb{C}^d$ to the unique node and A to the unique edge.
(b) The singular vectors of a matrix $A : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ arise from the representation of the directed cycle of length 2, with A assigned to one edge and $A^\top$ assigned to the other.
(c) The generalised eigenvectors of a pair of matrices $A, B : \mathbb{C}^d \to \mathbb{C}^d$ – i.e., non-zero solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \lambda \cdot B\boldsymbol{x}$ for some $\lambda \in \mathbb{C}$ – are the singular vectors of the representation of the Kronecker quiver with A on one edge and B on the other.

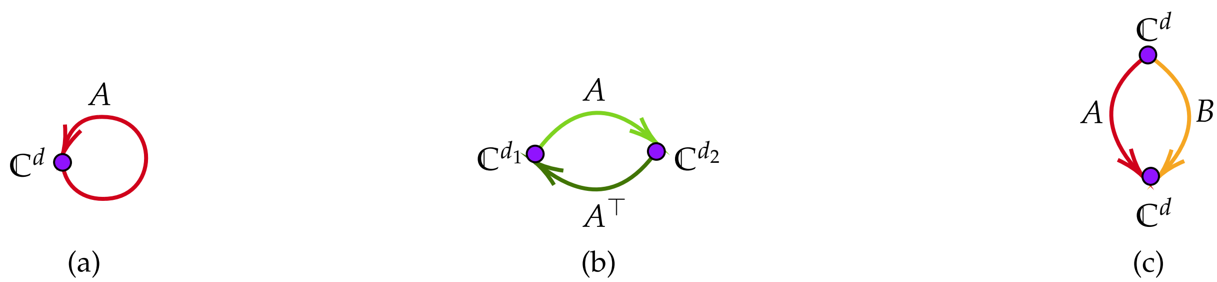

FIGURE 1. Quiver representations corresponding to (a) the eigenvectors of a matrix, (b) the singular vectors of a matrix, (c) the generalised eigenvectors of a pair of matrices.

For $d = d_1 = d_2$, a generic instance of any of these three quiver representations has d singular vectors.

Hyperquiver Representations and Tensors. This century has witnessed progress towards extending the spectral theory of matrices to the multilinear setting of tensors [34]. Given a tensor $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, we write $T(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, \cdot, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_n) \in \mathbb{C}^{d_j}$ for the vector with i-th coordinate

$$\sum_{i_1=1}^{d_1} \cdots \sum_{i_{j-1}=1}^{d_{j-1}} \sum_{i_{j+1}=1}^{d_{j+1}} \cdots \sum_{i_n=1}^{d_n} T_{i_1,\ldots,i_{j-1},i,i_{j+1},\ldots,i_n} (\boldsymbol{x}_1)_{i_1} \cdots (\boldsymbol{x}_{j-1})_{i_{j-1}} (\boldsymbol{x}_{j+1})_{i_{j+1}} \cdots (\boldsymbol{x}_n)_{i_n}.$$

Eigenvectors and singular vectors of tensors were introduced in [31, 33]. The eigenvectors of $T \in (\mathbb{C}^d)^{\otimes n}$ are all non-zero $\boldsymbol{x} \in \mathbb{C}^d$ satisfying

$$T(\,\cdot\,, \boldsymbol{x}, \ldots, \boldsymbol{x}) = \lambda \cdot \boldsymbol{x},$$

for some scalar $\lambda \in \mathbb{C}$. In the special case of matrices, this reduces to the familiar formula $A\boldsymbol{x} = \lambda \boldsymbol{x}$. Similarly, the singular vectors of a tensor $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ are the tuples of non-zero vectors $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}$ satisfying

$$T(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{j-1}, \cdot, \boldsymbol{x}_{j+1}, \dots, \boldsymbol{x}_n) = \mu_j \boldsymbol{x}_j$$

for all j. This specialises for matrices to the familiar pair of conditions $A\boldsymbol{x}_2 = \mu_1 \boldsymbol{x}_1$ and $A^\top \boldsymbol{x}_1 = \mu_2 \boldsymbol{x}_2$.

Eigenvectors and singular vectors have been computed for special classes of tensors in [35, 36]; they have been used to study hypergraphs [5, 34] and to learn parameters in latent variable models [3, 4]. For a symmetric tensor $T \in (\mathbb{C}^d)^{\otimes n}$ the eigenvectors are, equivalently, all non-zero $\boldsymbol{x} \in \mathbb{C}^d$ for which a scalar multiple $\lambda \cdot \boldsymbol{x}^{\otimes n}$ constitutes a critical point to the best rank-one approximation problem for T. Similarly, the singular vectors of $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ are all tuples of non-zero vectors $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}$ for which $\lambda \cdot \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n$ is a critical point to the best rank one approximation for T [31].

In order to create the appropriate generalisation of quiver singular vectors to subsume these notions from the spectral theory of tensors, we generalise from quivers to *hyperquivers*. In general, hyperquivers are obtained from quivers by allowing the source and target maps $s, t : E \to V$ to be multivalued. For our purposes, it suffices to consider hyperquivers where each edge $e \in E$ has a single target vertex. The hyperedge e now has a tuple of sources $(s_1(e), s_2(e), \dots, s_\mu(e)) \in V^\mu$ for some e-dependent integer $\mu \geq 1$. A representation $\boldsymbol{R} = (\boldsymbol{d}, T)$ of such a hyperquiver assigns to each vertex i a vector space $\mathbb{C}^{d_i}$ and to each edge e a tensor

$$T_e \in \mathbb{C}^{d_{t(e)}} \otimes \mathbb{C}^{d_{s_1(e)}} \otimes \dots \otimes \mathbb{C}^{d_{s_\mu(e)}}.$$

We identify a vector space $\mathbb{C}^d$ with its dual $(\mathbb{C}^d)^*$, allowing us to view the tensor T_e as a multilinear map

$$T_e : \prod_{j=1}^{\mu} \mathbb{C}^{d_{s_j(e)}} \to \mathbb{C}^{d_{t(e)}}$$

$$(\boldsymbol{x}_{s_1(e)}, \dots, \boldsymbol{x}_{s_\mu(e)}) \mapsto T_e(\cdot, \boldsymbol{x}_{s_1(e)}, \dots, \boldsymbol{x}_{s_\mu(e)}).$$

Our hyperquiver representations reduce to usual quiver representations when each edge has $\mu = 1$.

The set of singular vectors of a hyperquiver representation $\boldsymbol{R}$, denoted $\mathcal{S}(\boldsymbol{R})$, consists of all tuples $\big([\boldsymbol{x}_i] \in \mathbb{P}(\mathbb{C}^{d_i}) \mid i \in V\big)$ that satisfy

$$T_e(\cdot, \boldsymbol{x}_{i_1}, \dots, \boldsymbol{x}_{i_\mu}) = \lambda_e \cdot \boldsymbol{x}_j, \tag{1.1}$$

for some scalar λ_e, across every edge $e \in E$ of the form $(i_1, \dots, i_\mu) \to j$. We work with vectors in a product of projective spaces, since we require the vectors to be non-zero (as for the singular vectors of a matrix) and moreover because the equation (1.1) still holds after non-zero rescaling of each $\boldsymbol{x}_i$, albeit for different scalars λ_e.

Perhaps the simplest nontrivial family of examples is furnished by starting with the quiver with a single vertex and a single hyperedge with $m-1$ source vertices — we call this the m-Jordan hyperquiver. Consider the representation that assigns, to the vertex, the vector space $\mathbb{C}^d$ for some dimension $d \geq 0$, and to the edge, a tensor $T \in (\mathbb{C}^d)^{\otimes m}$, seen as a multilinear map $T : (\mathbb{C}^d)^{(m-1)} \to \mathbb{C}^d$ that contracts vectors along the last $m-1$ modes of T; see Figure 2a for the case $m = 3$. The singular vectors of this representation are all

$[\boldsymbol{x}] \in \mathbb{P}(\mathbb{C}^d)$ satisfying $T(\,\cdot\,, \boldsymbol{x}, \boldsymbol{x}, \dots, \boldsymbol{x}) = \lambda \cdot \boldsymbol{x}$ for some scalar $\lambda \in \mathbb{C}$. The singular vectors of the representation are therefore the eigenvectors of the tensor T.

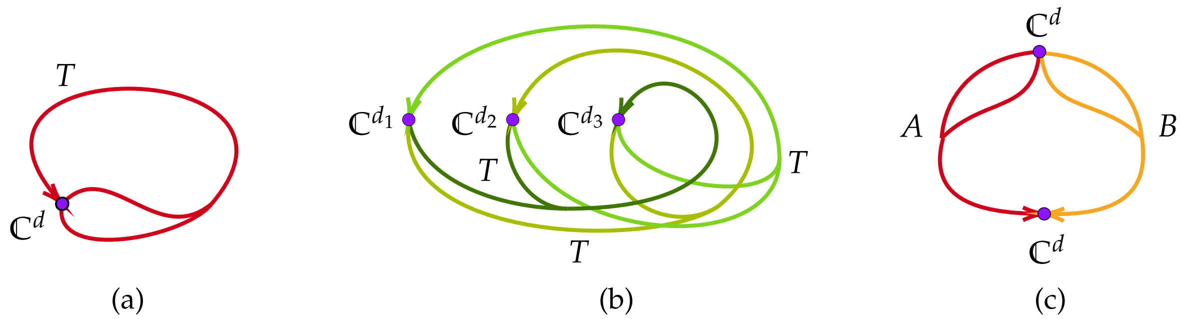

FIGURE 2. Examples of hyperquiver representations corresponding to (a) the eigenvectors of a tensor, (b) the singular vectors of a tensor, and (c) the generalised eigenvectors of a pair of tensors.

The compatibility conditions that define singular vectors can be reframed in terms of the vanishing of minors of suitable $d_i \times 2$ matrices. Hence $\mathcal{S}(\boldsymbol{R})$ is a multiprojective variety in $\prod_{i \in V} \mathbb{P}(\mathbb{C}^{d_i})$. This variety simultaneously forms a multilinear (and projective) generalisation of the linear *space of sections* of a quiver representation from [37], and a multi-tensor generalisation of the set of singular vectors of a single tensor from [20]. The property that a point lies in $\mathcal{S}(\boldsymbol{R})$ is equivariant under an orthogonal change of basis on each vector space, as is true for the singular vectors of a matrix, as follows. Let $([x_1], \dots, [x_n]) \in \prod_{i \in V} \mathbb{P}(\mathbb{C}^{d_i})$ be a singular vector tuple of a hyperquiver representation with tensors $T_e \in \mathbb{C}^{d_{t(e)}} \otimes \mathbb{C}^{d_{s_1(e)}} \otimes \dots \otimes \mathbb{C}^{d_{s_\mu(e)}}$ and let $Q_1, \dots, Q_n$ be a tuple of complex orthogonal matrices; i.e., $Q_i^\top Q_i = I_{d_i}$. Then $([Q_1 x_1], \dots, [Q_n x_n])$ is a singular vector tuple of the hyperquiver representation where each T_e has its source components multiplied by $Q_{s_j(e)}^\top$ and target component multiplied by $Q_{t(e)}$. We expect the topology of this variety, particularly its (co)homology groups, to provide rich and interesting isomorphism invariants for hyperquiver representations.

Main Result. We derive exact and explicit formulas for the *dimension* and *degree* of $\mathcal{S}(\boldsymbol{R})$ when $\boldsymbol{R}$ is a sufficiently generic representation of a given hyperquiver. Here is a simplified version, in the special case when all vector spaces are of the same dimension.

THEOREM. *Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a generic representation of a hyperquiver $H = (V, E)$ with $\boldsymbol{d} = (d, d, \dots, d)$. Let $N = (d-1)(|V| - |E|)$ and D be the coefficient of $\left(\prod_{i \in V} h_i\right)^{d-1}$ in the polynomial*

$$\left(\sum_{i \in V} h_i\right)^N \cdot \prod_{e \in E} \left(\sum_{k=1}^{d} h_{t(e)}^{k-1} \cdot h_{s(e)}^{d-k}\right), \quad \text{where} \quad h_{s(e)} := \sum_{j=1}^{\mu(e)} h_{s_j(e)}.$$

Then $\mathcal{S}(\boldsymbol{R}) = \varnothing$ if and only if $D = 0$. Otherwise, $\mathcal{S}(\boldsymbol{R})$ has dimension N and degree D. Moreover, if $\dim \mathcal{S}(\boldsymbol{R}) = 0$, then each singular vector tuple occurs with multiplicity 1.

EXAMPLE 1.1. Let $\boldsymbol{R}$ be the hyperquiver representation in Figure 3, with $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ a generic tensor. We have $N = (3-1)(2-2) = 0$. We seek the coefficient D of the monomial $h_1^2 h_2^2$ in the polynomial

$$\left((h_1+h_2)^2 + h_1(h_1+h_2) + h_1^2\right)^2 = 9h_1^4 + 18h_1^3h_2 + 15h_1^2h_2^2 + 6h_1h_2^3 + h_2^4.$$

We see that $D = 15$. Hence the singular vector variety $\mathcal{S}(\boldsymbol{R})$ has dimension $N = 0$ and consists of 15 singular vector tuples, each occurring with multiplicity 1.

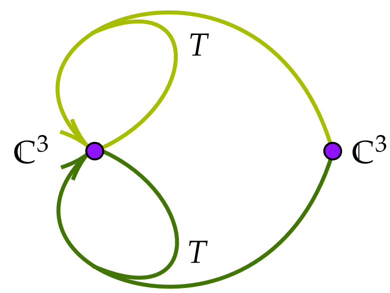

FIGURE 3. The light-green hyperedge is the contraction $T(\,\cdot\,, \boldsymbol{x}, \boldsymbol{y})$ and the dark-green hyperedge is the contraction $T(\boldsymbol{x}, \,\cdot\,, \boldsymbol{y})$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^3$ are on the left and right vertices respectively.

Our argument follows the work of Friedland and Ottaviani from [20] — we first construct a vector bundle whose generic global sections have the singular vectors of $\boldsymbol{R}$ as their zero set, and then apply a variant of Bertini's theorem to count singular vectors by computing the top Chern class of the bundle. The authors of [20] compute the number of singular vectors of a single generic tensor — this corresponds to counting the singular vectors of the hyperquiver representation depicted in Figure 2(b). Here we derive general formulas to describe the algebraic variety of singular vectors for an arbitrary network of (sufficiently generic) tensors.

Related Work. Special cases of our degree formula, all in the case $\dim \mathcal{S}(\boldsymbol{R}) = 0$, recover existing results from the literature — see [9] and [19, Corollary 3.2] for eigenvector counts, [20] for singular vector counts, and [13, 20] for generalised eigenvector counts. Other recent work that builds upon the approach in [20] includes [15, 38] which study the span of the singular vector tuples, [40] which studies tensors determined by their singular vectors, and the current work [2] which uses a related setup to count totally mixed Nash equilibria. The eigenscheme of a matrix [1] and ternary tensor [6] is a scheme-theoretic version of $\mathcal{S}(\boldsymbol{R})$ for the Jordan quiver in Figure 1a and the hyperquiver in Figure 2a.

Outline. The rest of this paper is organised as follows. In Section 2 we introduce hyperquiver representations and their singular vector varieties. We state our main result, Theorem 3.1, in Section 3 and describe a few of its applications. The construction of the vector bundle corresponding to a hyperquiver representation is given in Section 4, and our Bertini-type theorem – which we hope will be of independent interest – is proved in Section 5. We show that for generic $\boldsymbol{R}$ the hypotheses of the Bertini theorem are satisfied by the associated vector bundle in Section 6, and compute its top Chern class in Section 7. For completeness, we have collected relevant results from intersection theory in Appendix A.

2. The singular vector variety

We establish notation for hyperquiver representations, define their singular vector varieties, and highlight the genericity condition which plays a key role in the sequel. Without loss of generality, we henceforth assume $V = [n]$, where $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

DEFINITION 2.1. A *hyperquiver* $H = (V, E)$ consists of a finite set of *vertices* V of size $|V| = n$ and a finite set of *hyperedges* E. For each hyperedge $e \in E$ we have

(i) an integer $\mu(e) \geq 1$ called the *index* of e
(ii) a tuple of vertices $v(e) \in V^m$ called the *vertices* of e, where $m := \mu(e) + 1$.

For brevity, we may refer to a hyperedge as an edge and write μ as a shorthand for $\mu(e)$. The j-th entry of tuple $v(e)$ is denoted $s_j(e) \in V$. The tuple $s(e) := (s_1(e), \ldots, s_\mu(e))$ are the *sources* of e, and the vertex $t(e) := s_m(e)$ is the *target* of e.

REMARK 2.2. Usual quivers are the special case with $m = 2$ for all $e \in E$. Definition 2.1 does not exclude entries of $s(e)$ being equal to $t(e)$, nor does it exclude multiple hyperedges with the same tuple $v(e)$.

We now define representations of hyperquivers. The definition works for vector spaces over any field, but we focus on $\mathbb{C}$.

DEFINITION 2.3. Fix a hyperquiver $H = (V, E)$. Let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be a *dimension vector*. A *representation* $\boldsymbol{R} = (\boldsymbol{d}, T)$ of H assigns

(i) A vector space $\mathbb{C}^{d_i}$ to each vertex $i \in V$.
(ii) A tensor $T_e \in \mathbb{C}^e$ to each hyperedge $e \in E$, where $\mathbb{C}^e := \mathbb{C}^{d_{t(e)}} \otimes \mathbb{C}^{d_{s_1(e)}} \otimes \cdots \otimes \mathbb{C}^{d_{s_\mu(e)}}$, which is viewed as a multilinear map $\prod_{j=1}^{\mu} \mathbb{C}^{d_{s_j(e)}} \to \mathbb{C}^{d_{t(e)}}$.

We define for brevity

$$T_e(\boldsymbol{x}_{s(e)}) := T_e(\,\cdot\,, \boldsymbol{x}_{s_1(e)}, \ldots, \boldsymbol{x}_{s_\mu(e)}). \tag{2.1}$$

We say that two tensors T_e and $T_{e'}$ *agree up to permutation* if the tuples $v(e)$ and $v(e')$ agree up to a permutation σ and

$$(T_e)_{i_m, i_1, \ldots, i_{m-1}} = (T_{e'})_{i_{\sigma(m)}, i_{\sigma(1)}, \ldots, i_{\sigma(m-1)}}.$$

DEFINITION 2.4. The *singular vector variety* $\mathcal{S}(\boldsymbol{R})$ of a representation $\boldsymbol{R}$ consists of tuples $\chi = ([\boldsymbol{x}_1], \ldots, [\boldsymbol{x}_n]) \in \prod_{i=1}^n \mathbb{P}(\mathbb{C}^{d_i})$ such that

$$T_e(\boldsymbol{x}_{s(e)}) = \lambda_e \boldsymbol{x}_{t(e)}, \tag{2.2}$$

for some scalar $\lambda_e \in \mathbb{C}$, for every edge $e \in E$. The points of the variety are called the *singular vector tuples* of $\boldsymbol{R}$.

REMARK 2.5. The scalars λ_e in (2.2) can be thought of as the singular values of the singular vector tuple $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$. However, the non-homogeneity of (2.2) means that rescaling vectors in the tuple can change the singular values. We say that a singular vector tuple has a *singular value zero* if $\lambda_e = 0$ for some edge $e \in E$.

The singular vector variety is a subvariety of the multiprojective space $X = \prod_{i=1}^n \mathbb{P}(\mathbb{C}^{d_i})$. Its defining equations are as follows. The singular vector tuples $\boldsymbol{\chi} = ([\boldsymbol{x}_1], \dots, [\boldsymbol{x}_n])$ are the tuples whose $d_{t(e)} \times 2$ matrix

$$M_e(\boldsymbol{x}) := \begin{pmatrix} | & | \\ T_e(\boldsymbol{x}_{s(e)}) & \boldsymbol{x}_{t(e)} \\ | & | \end{pmatrix}$$

has rank ≤ 1 for all $e \in E$. The rank of this matrix depends only on the points $[\boldsymbol{x}_i] \in \mathbb{P}(\mathbb{C}^{d_i})$, and not on the vectors $\boldsymbol{x}_i \in \mathbb{C}^{d_i}$. Equations for the multiprojective variety $\mathcal{S}(\boldsymbol{R})$ are the 2×2 minors of all matrices $M_e(\boldsymbol{x})$ for $e \in E$. When we speak of the degree of $\mathcal{S}(\boldsymbol{R})$, we refer to the degree of its image under the Segre embedding $s : X \hookrightarrow \mathbb{P}^D$, for $D = \prod_{i=1}^n d_i - 1$.

Our main result finds the dimension and degree of the singular vector variety for a hyperquiver representation with sufficiently general tensors on the hyperedges. We say that a property P holds for a generic point of an affine variety V if there exists a Zariski-open set U in V such that P holds for all points in U. We call any point of such a U a *generic point* of V. One way that a hyperquiver representation can be sufficiently generic is for the tuple of tensors $(T_e \mid e \in E)$ assigned to its edges to be generic; that is, a generic point of $\prod_{e \in E} \otimes_{i=1}^m \mathbb{C}^{d_{s_i(e)}}$. This holds, for example, in Figure 1a and 1c. But our notion of genericity allows tensors on different hyperedges to coincide, as in Figure 1b. Our genericity condition is encoded by a partition of the hyperedges.

DEFINITION 2.6 (Genericity of a hyperquiver representation).

(i) A *partition* of a hyperquiver $H = (V, E)$ is a partition of its hyperedges $E = \coprod_{r=1}^M E_r$ such that for any hyperedges $e, e', e'' \in E_r$,
  (a) the indices $\mu(e)$ and $\mu(e')$ equal the same number μ
  (b) the tuples $v(e)$ and $v(e')$ coincide up to a permutation σ of $[\mu + 1]$
  (c) if σ and σ' are permutations in (b) for the pairs $v(e)$,$v(e')$ and $v(e')$,$v(e'')$ respectively, where $e \neq e'$ and $e' \neq e''$, then $\sigma(\mu + 1) \neq \sigma'(\mu + 1)$.

(ii) The *partition* of a representation $\boldsymbol{R} = (\boldsymbol{d}, T)$ is the unique partition of hyperedges such that for any $e, e' \in E_r$, the tensors on e and e' agree up to a permutation σ.

(iii) The representation $\boldsymbol{R} = (\boldsymbol{d}, T)$ is *generic* if given hyperedges $e_r \in E_r$ for $r \in [M]$, the tuple of tensors $(T_{e_1}, T_{e_2}, \dots, T_{e_M})$ is a generic point in $\prod_{r=1}^M \mathbb{C}^{e_r}$

EXAMPLE 2.7. We fix a basis on each vector space $U_i \cong \mathbb{C}^{d_i}$ in Definition 2.3 because being a singular vector tuple is not invariant under an arbitrary change of basis. For example, the quiver in Figure 1(b) with a generic square matrix $A : \mathbb{C}^d \to \mathbb{C}^d$ has d singular vector pairs $([\boldsymbol{x}], [\boldsymbol{y}])$. However, there exist change of basis matrices $M_1, M_2 \in GL(d, \mathbb{C})$ such that $M_2 A M_1^{-1} = I_d$, and the identity matrix I_d has infinitely many singular vector pairs: all pairs $([\boldsymbol{z}], [\boldsymbol{z}])$. The property of being a singular vector tuple is preserved, however, by an orthogonal change of basis, cf. the discussion in the introduction and [6, Remark 1.1].

REMARK 2.8. A (usual) quiver representation may be defined as assigning (abstract) vector spaces to vertices and linear maps to edges. Similarly, we could define a hyperquiver representation as assigning vector spaces U_i to each vertex i and multilinear maps $\mathcal{T}_e : \prod_{i=1}^{\mu} U_{s_j(e)} \to U_{t(e)}$ to each edge $e \in E$. The dimension of the linear space of sections of a quiver representation [37] and the dimension and degree of the singular vector variety of a hyperquiver representation are invariant under the action of $GL(U_i)$ on each vertex and edge. Since there is no given choice of a basis, or more generally no inner product on each vector space, the notion of a transpose of a linear map or permutation of a multilinear map does not make sense. Therefore, a generic representation in the sense of Definition 2.6 can only apply when each E_r is a singleton and we assign a distinct generic matrix or tensor to each edge. With a choice of basis, our genericity conditions allow permutations of tensors along the edges, via coarser partitions. The space of sections and the singular vector variety are then $O(d_i)$-invariant but not $GL(d_i)$-invariant.

3. Main theorem and its consequences

In this section, we present our main result in full generality and study its consequences.

THEOREM 3.1. *Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a generic hyperquiver representation and $\mathcal{S}(\boldsymbol{R})$ be the singular vector variety of $\boldsymbol{R}$. Let $N = \sum_{i\in V}(d_i - 1) - \sum_{e\in E}(d_{t(e)} - 1)$ and D be the coefficient of the monomial $h_1^{d_1-1} \cdots h_n^{d_n-1}$ in the polynomial*

$$\left(\sum_{i\in V} h_i\right)^N \cdot \prod_{e\in E} \left(\sum_{k=1}^{d_{t(e)}} h_{t(e)}^{k-1} h_{s(e)}^{d_{t(e)}-k}\right), \quad \text{where} \quad h_{s(e)} := \sum_{j=1}^{\mu(e)} h_{s_j(e)}. \tag{3.1}$$

Then $\mathcal{S}(\boldsymbol{R}) = \varnothing$ if and only if $D = 0$. Otherwise, $\mathcal{S}(\boldsymbol{R})$ is of pure dimension N and has degree D. If $\boldsymbol{R}$ has finitely many singular vector tuples, then each singular vector tuple is of multiplicity 1, is not isotropic, and has no singular value equal to 0.

Note that the partition from Definition 2.6 does not appear in the statement of Theorem 3.1: the partition provides a genericity condition for the result to hold, but the dimension and degree of the singular vector variety do not depend on the partition. Next we give a sufficient condition for a hyperquiver representation to consist of finitely many points. This condition applies to Figure 2a and Figure 2b, but not to Figure 2c.

COROLLARY 3.2. *The hyperquivers with finitely many singular vector tuples for any choice of generic representation are those whose vertices each have exactly one incoming hyperedge.*

PROOF. If $\dim \mathcal{S}(\boldsymbol{R}) = N = \sum_{i\in V}(d_i - 1) - \sum_{e\in E}(d_{t(e)} - 1) = 0$ for all dimensions d_i, then $\sum_{i\in V}(d_i - 1) = \sum_{e\in E}(d_{t(e)} - 1)$ as polynomials in the variables d_i. Each d_i appears exactly once on the left hand side of the equation. Hence it must also appear exactly once on the right hand side. Therefore $|V| = |E|$ and every $i \in V$ has exactly one $e \in E$ with $i = t(e)$. □

We show how Theorem 3.1 specialises to count the eigenvectors and singular vectors of a generic tensor, as well as to count the solutions to the generalised eigenproblem from [13].

EXAMPLE 3.3 (Eigenvectors of a tensor). We continue our discussion from the introduction. The representation of the m-Jordan hyperquiver with a generic tensor $T \in (\mathbb{C}^d)^{\otimes m}$ on its hyperedge is generic in the sense of Definition 2.6, since we have only one hyperedge. There are finitely many eigenvectors, by Corollary 3.2. The polynomial (3.1) is

$$\sum_{k=1}^{d} h^{k-1}((m-1)h)^{d-k} = \left(\sum_{k=1}^{d}(m-1)^{d-k}\right) h^{d-1} = \frac{(m-1)^d - 1}{m-2} h^{d-1}.$$

The coefficient of h^{d-1} is $\frac{(m-1)^d-1}{m-2}$. This agrees with the count for the number of eigenvectors of a generic tensor from [9, Theorem 1.2] and [19, Corollary 3.2].

We now consider singular vectors. A result of Friedland and Ottaviani [20] is:

THEOREM 3.4 (Friedland and Ottaviani [20, Theorem 1]). *The number of singular vectors of a generic tensor $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ is the coefficient of the monomial $h_1^{d_1-1} \ldots h_n^{d_n-1}$ in the polynomial*

$$\prod_{i\in[n]} \frac{\widehat{h}_i^{d_i} - h_i^{d_i}}{\widehat{h}_i - h_i}, \qquad \textit{where} \quad \widehat{h}_i := \sum_{j\in[n]\setminus\{i\}} h_j,\ i \in [n]. \tag{3.2}$$

Each singular vector tuple is of multiplicity 1, is not isotropic, and does not have singular value 0.

We explain how the above result follows from Theorem 3.1.

EXAMPLE 3.5 (Singular vectors of a tensor). Consider the hyperquiver with n vertices $V = [n]$ and n hyperedges. For every vertex $i \in V$, there is a hyperedge e_i with $s(e_i) = (1, \ldots, i-1, i+1, \ldots, n)$ and target $t(e) = i$. Consider the representation that assigns the vector space $\mathbb{C}^{d_i}$ to each vertex and the same generic tensor $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ to each hyperedge. On each edge e_i, the tensor T is seen as a multilinear map

$$\begin{aligned} T : \mathbb{C}^{d_1} \times \ldots \times \mathbb{C}^{d_{i-1}} \times \mathbb{C}^{d_{i+1}} \times \cdots \times \mathbb{C}^{d_n} &\to \mathbb{C}^{d_i} \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n) &\mapsto T(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \cdot, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n). \end{aligned}$$

This representation is generic in the sense of Definition 2.6, where the partition of the edge set E has size $M = 1$ and the permutation σ sending $v(e_i)$ to $v(e_j)$ is the one that swaps i and j and keeps all other entries fixed. Figure 2b illustrates this representation for $n = 3$. The singular vector variety consists of all non-zero vectors $\boldsymbol{x}_i \in \mathbb{C}^{d_i}$ such that $T(\boldsymbol{x}_{s(e)}) = \lambda_e \boldsymbol{x}_{t(e)}$ for some $\lambda_e \in \mathbb{C}$ and all $e \in E$, where $T(\boldsymbol{x}_{s(e)})$ is defined in (2.1). That is, the singular vector variety consists of all singular vector tuples of T. Corollary 3.2 shows that there are finitely many singular vector tuples. The polynomial (3.1) specialises to

$$\prod_{i\in[n]} \left(\sum_{k=1}^{d_i} h_i^{k-1} \widehat{h}_i^{d_i-k}\right), \quad \text{where} \quad \widehat{h}_i := \sum_{j\in[n]\setminus\{i\}} h_j,\ i \in [n].$$

This is equivalent to (3.2) via the identity $\frac{x^n - y^n}{x-y} = \sum_{k=1}^{n} x^{k-1} y^{n-k}$.

EXAMPLE 3.6 (The generalised tensor eigenvalue problem). Consider a generic representation of the *Kronecker hyperquiver* with a generic pair of tensors $A, B \in \mathbb{C}^{d_2} \otimes (\mathbb{C}^{d_1})^{\otimes(m-1)}$, see Figure 2c with $m = 3$ and $d = d_1 = d_2$. The edge set E has a partition with $M = 2$.

We remark that Corollary 3.2 implies that there will not be finitely many singular vector tuples for *all* representations of this hyperquiver. There will be a non-zero finite number of singular vectors if and only if $d := d_1 = d_2$ since this is when $N = 0$ in Theorem 3.1. The singular vector tuples are the non-zero pairs $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^d$ such that $A(\cdot, \boldsymbol{x}, \ldots, \boldsymbol{x}) = \lambda' \boldsymbol{y}$ and $B(\cdot, \boldsymbol{x}, \ldots, \boldsymbol{x}) = \lambda'' \boldsymbol{y}$, for some $\lambda', \lambda'' \in \mathbb{C}$. This reduces to the single equation

$$A(\cdot, \boldsymbol{x}, \ldots, \boldsymbol{x}) = \lambda B(\cdot, \boldsymbol{x}, \ldots, \boldsymbol{x})$$

for some $\lambda \in \mathbb{C}$. This is a tensor-analogue of the generalised eigenvalue problem for two matrices. It was shown in [20, Corollary 16] and [13, Theorem 2.1] that there are $d(m-1)^{d-1}$ generalised tensor eigenvalue pairs $\boldsymbol{x}$ and $\boldsymbol{y}$ for the tensors A and B. Our general formula in Theorem 3.1 also recovers this number, as follows. The polynomial (3.1) is

$$\left(\sum_{k=1}^{d} h_2^{k-1} ((m-1)h_1)^{d-k} \right) \left(\sum_{\ell=1}^{d} h_2^{\ell-1} ((m-1)h_1)^{d-\ell} \right). \tag{3.3}$$

A monomial $h_1^{d-1} h_2^{d-1}$ is obtained from the product of a k-th summand and an ℓ-th summand such that $k + \ell = d + 1$. There are d such pairs of summands $k, \ell \in \{1, \ldots, d\}$. Each such monomial will have a coefficient of $(m-1)^{d-1}$. Hence the coefficient of $h_1^{d-1} h_2^{d-1}$ in (3.3) is $d(m-1)^{d-1}$.

Now we find the dimension and degree of the singular vector variety $\mathcal{S}(\boldsymbol{R})$ for a generic representation $\boldsymbol{R}$ of a hyperquiver with a single hyperedge, as shown in Figure 4.

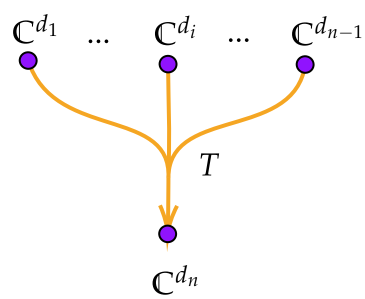

FIGURE 4. A hyperquiver with a single hyperedge and a representation

COROLLARY 3.7. *Let H be a hyperquiver with one hyperedge with all entries of its tuple of vertices distinct. Let $\boldsymbol{R}$ be the representation that assigns the vector space $\mathbb{C}^{d_i}$ to each vertex i and a generic tensor to the hyperedge. Then:*

(a) The dimension of $\mathcal{S}(\boldsymbol{R})$ is $N = \sum_{i=1}^{n-1} d_i - n + 1$
(b) The degree of $\mathcal{S}(\boldsymbol{R})$ is

$$\sum_{k=1}^{d_n} \sum_{\substack{k_1 + \cdots + k_{n-1} \\ = d_n - k}} \binom{d_n - k}{k_1, \ldots, k_{n-1}} \binom{N}{d_1 - 1 - k_1, \ldots, d_{n-1} - 1 - k_{n-1}, d_n - k}. \tag{3.4}$$

PROOF. The dimension of $\mathcal{S}(\boldsymbol{R})$ is $N = (\sum_{i=1}^{n} d_i - n) - (d_n - 1) = \sum_{i=1}^{n-1} d_i - n + 1$, by Theorem 3.1. The degree of $\mathcal{S}(\boldsymbol{R})$ is the coefficient of $h_1^{d_1-1} \cdots h_n^{d_n-1}$ in the product

$$\underbrace{\left(\sum_{i=1}^{n} h_i\right)^N}_{(1)} \underbrace{\left(\sum_{k=1}^{d_n} \left(\sum_{i=1}^{n-1} h_i\right)^{d_n-k} h_n^{k-1}\right)}_{(2)}.$$

For each $k \in \{1, \ldots, d_n\}$, the monomial $h_1^{k_1} \cdots h_{n-1}^{k_{n-1}} h_n^{k-1}$ in the expansion of (2) for some $k_1, \ldots, k_{n-1}$ such that $\sum_{i=1}^{n-1} k_i = d_n - k$ has coefficient $\binom{d_n-k}{k_1,\ldots,k_{n-1}}$. This is combined with the monomial $h_1^{d_1-1-k_1} \cdots h_{n-1}^{d_n-1-k_{n-1}} h_n^{d_n-k}$ from the expansion of (1), which has coefficient $\binom{N}{d_1-1-k_1,\ldots,d_{n-1}-1-k_{n-1},d_n-k}$. Multiplying these coefficients and summing over those $k_1, \ldots, k_{n-1}$ with $\sum_{i=1}^{n-1} k_i = d_n - k$, we obtain

$$\sum_{\substack{k_1+\cdots+k_{n-1}\\ =d_n-k}} \binom{d_n-k}{k_1,\ldots,k_{n-1}} \binom{N}{d_1-1-k_1,\ldots,d_{n-1}-1-k_{n-1},d_n-k}.$$

Summing over $k = 1, \ldots, d_n$ gives the result. □

When $d := d_1 = \cdots = d_n$, we can use Corollary 3.7 to find the degree of $\mathcal{S}(\boldsymbol{R})$, which is displayed in Table 1 for $d = 1, \ldots, 6$ and $n = 2, \ldots, 6$. Observe that: (i) the degree row of $d = 2$ consist of the factorial numbers; and (ii) the degree column of $n = 2$ consist of powers of 2. We explain these observations. To see (i), if $d = 2$, then (3.4) becomes

$$\sum_{k=1}^{2} \sum_{\substack{k_1+\ldots+k_{n-1}\\ =2-k}} \binom{2-k}{k_1,\ldots,k_{n-1}} \binom{n-1}{1-k_1,\ldots,1-k_{n-1},2-k}. \tag{3.5}$$

When $k = 2$, the only summands satisfying $k_1 + \cdots + k_{n-1} = 2 - k$ is $k_1 = \cdots = k_{n-1} = 0$, which is 1 for the first factor and $(n-1)!$ for the second factor in (3.5). When $k = 1$, the only allowed indices are of the form $k_i = 1$ and $k_j = 0$ for all $i \neq j$, from which we get 1 for the first factor and $(n-1)!$ for the second factor in (3.5). Since there are $n - 1$ such allowed indices, (3.5) evaluates to $(n-1)! + (n-1)(n-1)! = n!$. For (ii), when $n = 2$, we have

$$\begin{aligned}\sum_{k=1}^{d} \sum_{k_1=d-k} \binom{d-k}{k_1} \binom{d-1}{d-1-k_1, d-k} &= \sum_{k=1}^{d} \binom{d-k}{d-k} \binom{d-1}{k-1, d-k} \\ &= \sum_{k=0}^{d-1} \binom{d-1}{k, d-1-k} = \sum_{k=0}^{d-1} \binom{d-1}{k} = 2^{d-1}.\end{aligned}$$

EXAMPLE 3.8 (Periodic orbits of order n). Consider the hyperquiver representation in Figure 5 with a generic tensor $T \in (\mathbb{C}^d)^{\otimes m}$. The singular vector tuples are the non-zero

$d \backslash n$	2	3	4	5	6
1	1	1	1	1	1
2	2	6	24	120	720
3	4	66	1980	93240	6350400
4	8	840	218400	110510000	96864800000
5	16	11410	27512100	1.5873×10^{11}	1.89313×10^{15}
6	32	160776	3741400000	2.54601×10^{14}	4.26416×10^{19}

TABLE 1. The degree of the singular vector variety $\mathcal{S}(\boldsymbol{R})$ of the hyperquiver in Figure 4 with $d_1 = ... = d_n = d$ and generic tensor T. The dimension of $\mathcal{S}(\boldsymbol{R})$ is $N = (d-1)(n-1)$. In particular, $\mathcal{S}(\boldsymbol{R})$ is positive-dimensional except in the first row.

vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{C}^d$ such that

$$
\begin{aligned}
T(\,\cdot\,, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_1) &= \lambda_1 \boldsymbol{x}_2 \\
T(\,\cdot\,, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_2) &= \lambda_2 \boldsymbol{x}_3 \\
&\vdots \\
T(\,\cdot\,, \boldsymbol{x}_n, \ldots, \boldsymbol{x}_n) &= \lambda_n \boldsymbol{x}_1
\end{aligned}
$$

for some $\lambda_i \in \mathbb{C}$. In other words, each $\boldsymbol{x}_i$ is a periodic point of order n.

The hyperquiver representation is not generic in the sense of Definition 2.6 as edges with different tuples $v(e)$ up to permutation are assigned the same tensor T. Hence Theorem 3.1 does not apply. Nonetheless, we predict the dimension and degree, using Theroem 3.1. The result predicts finitely many n-periodic points, by Corollary 3.2. Their count is predicted to be the coefficient of the monomial $h_1^{d-1} \ldots h_n^{d-1}$ in the polynomial

$$
\left(\sum_{k=1}^{d} h_2^{k-1} (\mu h_1)^{d-k} \right) \left(\sum_{k=1}^{d} h_3^{k-1} (\mu h_2)^{d-k} \right) \ldots \left(\sum_{k=1}^{d} h_1^{k-1} (\mu h_n)^{d-k} \right), \tag{3.6}
$$

by Theorem 3.1. This monomial is obtained from the product of terms

$$
(h_2^{k-1} (\mu h_1)^{d-k}) (h_3^{k-1} (\mu h_2)^{d-k}) \ldots (h_1^{k-1} (\mu h_n)^{d-k})
$$

coming from each of the respective factors in (3.6), for each $k \in [d]$. The coefficient of this product is $\mu^{n(d-k)}$. Thus, the coefficient of $h_1^{d-1} \ldots h_n^{d-1}$ in (3.6) is

$$
\sum_{k=1}^{d} \mu^{n(d-k)} = \frac{\mu^{nd} - 1}{\mu^n - 1} = \frac{(m-1)^{nd} - 1}{(m-1)^n - 1}.
$$

This turns out to be the correct number of period-n fixed points, as proved in [18, Corollary 3.2]. The number of eigenvectors of a generic tensor is the special case $n = 1$ (Example 3.3).

EXAMPLE 3.9 (Empty singular vector variety). Consider the quiver in Figure 6, where the vertices are assigned vector spaces of dimension $d > 1$, and the two edges are assigned generic matrices $A, B \in \mathbb{C}^{d \times d}$. Any singular vector would need to be an eigenvector of both

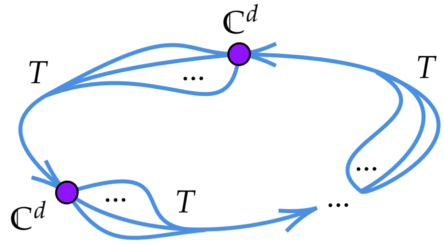

FIGURE 5. A hyperquiver representing a period-n orbit

matrices A and B, but a pair of generic matrices A and B do not share an eigenvector. We see how the emptiness of the singular vector variety is captured by Theorem 1: The polynomial is $(dh_1^{d-1})^2$, which has coefficient of $(h_1h_2)^{d-1}$ equal to zero.

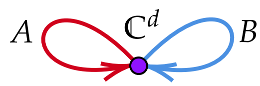

FIGURE 6. A quiver representation with empty singular vector variety

EXAMPLE 3.10 (Insufficiently generic representations). The quiver representations in Figure 7 with $d > 1$ and generic matrix $A \in \mathbb{C}^{d \times d}$ do not satisfy the genericity conditions in Definition 2.6. In Figure 7(a), the only permutations σ, σ' on $\{1, 2\}$ sending the matrix A on one edge to the matrix A on the other edge and vice versa are the identity permutations, which fail to satisfy the condition $\sigma(2) \neq \sigma'(2)$, causing one of the edges to be redundant. The resulting singular vector variety has dimension $d - 1$ and degree 2^{d-1} by Corollary 3.7, rather than the expected dimension 0 and degree d in Example 3.6. In Figure 7(b), the singular vectors are the non-zero points $\boldsymbol{x} \in \mathbb{C}^d$ such that $A^2\boldsymbol{x} = \lambda A\boldsymbol{x}$ for some $\lambda \in \mathbb{C}$, of which there are d solutions, rather than the expected 0 solutions in Theorem 3.1.

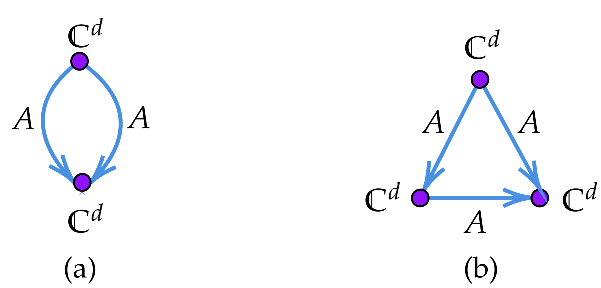

FIGURE 7. Insufficiently generic quiver representations

In the remainder of this section, we explore connections to dynamical systems and message passing.

EXAMPLE 3.11 (Fixed Homology Classes). A *parameterised dynamical system* is a continuous map $f : X \times P \to X$, where X and P are compact triangulable topological spaces, respectively called the *state* and *parameter* space of f. Taking homology with complex coefficients, we obtain a $\mathbb{C}$-linear map

$$H_k f : H_k(X \times P) \to H_k(X)$$

in each dimension $k \geq 0$. We know from the Künneth formula [39, Section 5.3] that the domain of $H_k f$ is naturally isomorphic to the direct sum $\bigoplus_{i+j=k} H_i(X) \otimes H_j(P)$. Therefore, each $H_k f$ admits a component of the form

$$T_k : H_k(X) \otimes H_0(P) \to H_k(X),$$

We say that a non-zero homology class $\xi \in H_k(X)$ is *fixed* by f at a non-zero homology class $\eta \in H_0(P)$ whenever there exists a scalar $\lambda \in \mathbb{C}$ satisfying $T_k(\xi \otimes \eta) = \lambda \cdot \xi$. The set of all such fixed homology classes (up to scaling) is the singular vector variety of the hyperquiver representation in Figure 8.

Let $k := \dim H_k(X)$ and suppose P has d connected components; i.e., $\dim H_0(P) = d$. Then the singular vector variety has dimension $d-1$ and degree equal to the coefficient of $h_1^{k-1}h_2^{d-1}$ in the polynomial $(h_1+h_2)^{d-1}\sum_{j=1}^{k}(h_1+h_2)^{k-j}h_2^{j-1}$, by Theorem 3.1. The monomial $h_1^{k-1}h_2^{d-1}$ arises by pairing a term $\binom{k-j}{i}h_1^i h_2^{k-j-i}h_2^{j-1} = \binom{k-j}{i}h_1^i h_2^{k-i-1}$ in the expanded sum with the term $\binom{d-1}{k-i-1}h_1^{k-i-1}h_2^{(d-1)-(k-i-1)}$ in the expanded parentheses, for all $0 \leq i \leq k-j$ and $1 \leq j \leq k$. Thus, its coefficient is

$$\sum_{j=1}^{k}\sum_{i=0}^{k-j}\binom{k-j}{i}\binom{d-1}{k-i-1}$$

In particular, if P is connected (i.e., $d = 1$), then there is exactly one non-zero homology class in $H_k(X)$ fixed by f.

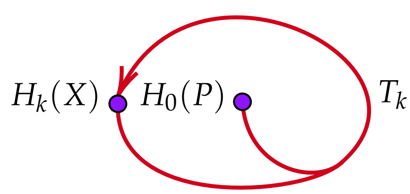

FIGURE 8. Fixed points in homology

EXAMPLE 3.12 (Message Passing). Our framework counts the fixed points of certain multilinear message passing operations, as we now describe. Assign vectors $\boldsymbol{x}_i^{(0)} := \boldsymbol{x}_i \in \mathbb{C}^{d_i}$ to each $i \in V$. Apply the multilinear map T_e to the vectors $(\boldsymbol{x}_{s_1(e)}^{(k)}, \ldots, \boldsymbol{x}_{s_\mu(e)}^{(k)})$ at nodes in $s(e)$. Then, update the vector at the target vertex $t(e)$ to

$$\boldsymbol{x}_{t(e)}^{(k+1)} := T_e(\boldsymbol{x}_{s(e)}^{(k)}) \in \mathbb{C}^{d_{t(e)}}. \tag{3.7}$$

In the limit, one converges to a fixed point of the update steps. The singular vector variety consists of tuples of directions in $\mathbb{C}^{d_i}$ that are fixed under these operations, for any order of update steps.

We compare the update (3.7) to message passing graph neural networks, see e.g. [23, 25]. The vector at each vertex is the features of the vertex. The vectors typically lie in a vector space of the same dimension, as in Theorem 1. Message passing operations take the form

$$\boldsymbol{x}_i^{(k+1)} = f(\{\boldsymbol{x}_i^{(k)}\} \cup \{\boldsymbol{x}_j^{(k)} : j \in \mathcal{N}(i)\}), \tag{3.8}$$

where $\mathcal{N}(i)$ is the neighbourhood of vertex i. That is, the vector of features at node i in the $(k+1)$-th step depends on the features of node i and its neighbours at the k-th step. Our update step in (3.7) is a special case of (3.8). We relate (3.7) to operations in the literature.

The function f in (3.8) often involves a non-linearity, applied pointwise. In comparison, we focus on the (multi)linear setting, as discussed for example in [11]. There, the authors study the optimisation landscapes of linear update steps, relating them to power iteration algorithms. Our approach to count the locus of fixed points sheds insight into the global structure of this optimisation landscape, in the spirit of [10, 14]. Studying such fixed point conditions directly is the starting point of implicit deep learning [17, 24].

The neighbourhood $\mathcal{N}(i)$, for us, consists of nodes j that appear in a tuple $s(e)$ for some edge e with $t(e) = i$. Update steps are usually over a graph rather than a hypergraph. The tensor multiplications from (3.7) incorporate higher-order interactions. Such higher-order structure also appears in tensorised graph neural networks [27] and message passing simplicial networks [8].

4. The singular vector bundle

In this section, we define the singular vector bundle. It is a vector bundle on $X = \prod_{i=1}^n \mathbb{P}(\mathbb{C}^{d_i})$ whose global sections are associated to hyperquiver representations. The zeros of a section are the singular vectors of the corresponding representation.

Following [20, Section 2], for each integer $d > 0$ we consider four vector bundles over $\mathbb{P}(\mathbb{C}^d)$: the *free* bundle $\mathscr{F}(d)$, the *tautological* bundle $\mathscr{T}(d)$, the *quotient* bundle $\mathscr{Q}(d)$, and the *hyperplane* bundle $\mathscr{H}(d)$. Their fibres at each $[\boldsymbol{x}] \in \mathbb{P}(\mathbb{C}^d)$ are

$$\mathscr{F}(d)_{[\boldsymbol{x}]} = \mathbb{C}^d \qquad \mathscr{Q}(d)_{[\boldsymbol{x}]} = \mathbb{C}^d / \operatorname{span}(\boldsymbol{x})$$
$$\mathscr{T}(d)_{[\boldsymbol{x}]} = \operatorname{span}(\boldsymbol{x}) \qquad \mathscr{H}(d)_{[\boldsymbol{x}]} = \operatorname{span}(\boldsymbol{x})^\vee.$$

Here if V is a vector space or vector bundle, then $V^\vee$ denotes its dual. We have a short exact sequence of vector bundles

$$0 \to \mathscr{T}(d_i) \to \mathscr{F}(d_i) \to \mathscr{Q}(d_i) \to 0. \tag{4.1}$$

There are projection maps $\pi_i : X \to \mathbb{P}(\mathbb{C}^{d_i})$ with $\pi_i(\chi) = [\boldsymbol{x}_i]$, where $\chi = ([\boldsymbol{x}_1], \ldots, [\boldsymbol{x}_n])$. We pull back a vector bundle $\mathscr{B}$ over $\mathbb{P}(\mathbb{C}^{d_i})$ to a bundle $\pi_i^*\mathscr{B}$ over X, whose fiber at $\chi \in X$ equals $\mathscr{B}_{[\boldsymbol{x}_i]}$. There is an exact sequence $0 \to \mathscr{T}(d_i)_{[\boldsymbol{x}_i]} \to \mathscr{F}(d_i)_{[\boldsymbol{x}_i]} \to \mathscr{Q}(d_i)_{[\boldsymbol{x}_i]} \to 0$ of vector spaces at every $[\boldsymbol{x}_i] \in \mathbb{P}(\mathbb{C}^{d_i})$. Hence there is an exact sequence of vector bundles

$$0 \to \pi_i^*\mathscr{T}(d_i) \to \pi_i^*\mathscr{F}(d_i) \to \pi_i^*\mathscr{Q}(d_i) \to 0. \tag{4.2}$$

DEFINITION 4.1. Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a hyperquiver representation and let $X = \prod_{i=1}^{n} \mathbb{P}(\mathbb{C}^{d_i})$. For each hyperedge $e \in E$, we consider the following vector bundles over X.

$$\mathscr{T}(e) := \bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{T}(d_{s_j(e)}), \qquad \mathscr{B}(e) := \operatorname{Hom}\left(\mathscr{T}(e), \pi_{t(e)}^* \mathscr{Q}(d_{t(e)})\right).$$

We define the *singular vector bundle* of $\boldsymbol{R}$ over X to be $\mathscr{B}(\boldsymbol{R}) := \bigoplus_{e \in E} \mathscr{B}(e)$.

The vector bundle $\mathscr{B}(\boldsymbol{R})$ depends on the hypergraph H and the assigned vector spaces U, but not on the multilinear maps T. It can be written in terms of a partition of edges as $\mathscr{B}(\boldsymbol{R}) = \bigoplus_{r=1}^{M} \bigoplus_{e \in E_r} \mathscr{B}(e)$. We will see that when $\boldsymbol{R}$ is a generic hyperquiver representation, the zero locus of a generic section of $\mathscr{B}(\boldsymbol{R})$ is the singular vector variety $\mathcal{S}(\boldsymbol{R})$. We make the following observations about its summands $\mathscr{B}(e)$.

PROPOSITION 4.2. *Let $\mathscr{B}(e) = \operatorname{Hom}\left(\mathscr{T}(e), \pi_{t(e)}^* \mathscr{Q}(d_{t(e)})\right)$. Then the following hold.*

(a) The fibre of $\mathscr{B}(e)$ at χ is $\operatorname{Hom}\left(\operatorname{span}\left(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}\right), \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)})\right)$.

(b) The bundle $\mathscr{B}(e)$ has rank $d_{t(e)} - 1$.

(c) We have the isomorphism $\mathscr{B}(e) = \left(\bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{H}(d_{s_j(e)})\right) \otimes \pi_{t(e)}^* \mathscr{Q}(d_{t(e)})$.

PROOF. The bundle $\mathscr{T}(e)$ has fibres

$$\begin{aligned}\mathscr{T}(e)_\chi &= \bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{T}(d_{s_j(e)})_\chi = \bigotimes_{j=1}^{\mu(e)} \mathscr{T}(d_{s_j(e)})_{[\boldsymbol{x}_{s_j(e)}]} \\ &= \bigotimes_{j=1}^{\mu(e)} \operatorname{span}(\boldsymbol{x}_{s_j(e)}) = \operatorname{span}\left(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}\right).\end{aligned}$$

The bundle $\pi_{t(e)}^* \mathscr{Q}(d_{t(e)})$ has fibre $\pi_{t(e)}^* \mathscr{Q}(d_{t(e)})_\chi = \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)})$. This proves (a). Then (b) follows, since the dimension of the fibre is $d_{t(e)} - 1$. To prove (c), observe that $\mathscr{B}(e) \simeq \mathscr{T}(e)^\vee \otimes \pi_{t(e)}^* \mathscr{Q}(d_{t(e)})$ and that

$$\begin{aligned}\mathscr{T}(e)^\vee = \left(\bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{T}(d_{s_j(e)})\right)^\vee &\simeq \bigotimes_{j=1}^{\mu(e)} \left(\pi_{s_j(e)}^* \mathscr{T}(d_{s_j(e)})\right)^\vee \\ &\simeq \bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{T}(d_{s_j(e)})^\vee = \bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^* \mathscr{H}(d_{s_j(e)}).\end{aligned}$$ □

We relate the singular vector variety to the singular vector bundle. The global sections of a vector bundle $\mathscr{B}$ are denoted by $\Gamma(\mathscr{B})$. They are the holomorphic maps $\sigma : X \to \mathscr{B}$ that send each $\chi \in X$ to a point in $\mathscr{B}_\chi$. A global section of $\mathscr{B}(e)$ is a map sending each $\chi \in X$ to an element in

$$\operatorname{Hom}\left(\operatorname{span}\left(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}\right), \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)})\right),$$

by Proposition 4.2(a). Definition 2.6(ii) of a partition gives an equivalence relation between tensors assigned to E_r via permutation of the modes. Following the notation of Definition

2.6(iii), we denote by $T_r \in \mathbb{C}^{e_r}$ a representative for the class corresponding to E_r, for some $e_r \in E_r$, and we define $T_r(\boldsymbol{x}_{s(e)}) := T_e(\boldsymbol{x}_{s(e)})$ for all $e \in E_r$, where $T_e(\boldsymbol{x}_{s(e)})$ is defined in (2.1). A tensor $T \in \mathbb{C}^{e_r}$ determines a global section of $\mathscr{B}(e)$ for every $e \in E_r$, which we denote by $L_e(T)$. The map $L_e(T)$ sends $\boldsymbol{\chi}$ to the map

$$\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)} \mapsto \overline{T(\boldsymbol{x}_{s(e)})} \in \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)}).$$

where $\overline{T(\boldsymbol{x}_{s(e)})}$ is the image of $T(\boldsymbol{x}_{s(e)})$ in the quotient vector space $\mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)})$. In other words, following [20, Lemma 9], we define the map

$$\begin{aligned} L_e : \mathbb{C}^{e_r} &\longrightarrow \Gamma(\mathscr{B}(e)) \\ T &\longmapsto L_e(T). \end{aligned}$$

We form the composite map

$$\begin{aligned} L : \bigoplus_{r=1}^{M} \mathbb{C}^{e_r} &\longrightarrow \Gamma(\mathscr{B}(\boldsymbol{R})) \\ (T_1, \ldots, T_M) &\longmapsto \bigoplus_{r=1}^{M} \bigoplus_{e \in E_r} L_e(T_r). \end{aligned} \tag{4.3}$$

We connect the global sections in the image of L to the singular vector tuples of a hyperquiver representation, generalizing [20, Lemma 11].

Proposition 4.3. *Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a hyperquiver representation. Let $X = \prod_{i=1}^{n} \mathbb{P}(\mathbb{C}^{d_i})$ and let $\mathscr{B}(\boldsymbol{R})$ be the singular vector bundle, with $L : \bigoplus_{r=1}^{M} \mathbb{C}^{e_r} \to \Gamma(\mathscr{B}(\boldsymbol{R}))$ the map in (4.3). Then a point $\boldsymbol{\chi} \in X$ lies in the zero locus of the section $\sigma = L((T_r)_{r=1}^{M})$ if and only if $\boldsymbol{\chi}$ is a singular vector tuple of $\boldsymbol{R}$.*

Proof. $L((T_r)_{r=1}^{M})(\boldsymbol{\chi})$ is the $|E|$-tuple of zero maps each in $\mathscr{B}(e)_{\boldsymbol{\chi}}$ if and only if for all $e \in E_r$ and $r \in [M]$, $L_e(T_r)(\boldsymbol{\chi})(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}) = \overline{0}$, if and only if $T_r(\boldsymbol{x}_{s(e)}) = \lambda_e \boldsymbol{x}_{t(e)}$ for some $\lambda_e \in \mathbb{C}$, if and only if $\boldsymbol{\chi}$ is a singular vector tuple of the hyperquiver representation $\boldsymbol{R}$. $\square$

In light of the preceding result, it becomes necessary to determine the image of L within $\Gamma(\mathscr{B}(\boldsymbol{R}))$. For this purpose, we make use of the following Künneth formula for vector bundles. Note that $H^0(X, \mathscr{B}) := \Gamma(\mathscr{B})$.

Proposition 4.4 (Künneth Formula, [29, Proposition 9.2.4]). *Let X and Y be complex varieties and $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection maps. If $\mathscr{F}$ and $\mathscr{G}$ are vector bundles on X and Y respectively, then*

$$H^n(X \times Y, \pi_X^* \mathscr{F} \otimes \pi_Y^* \mathscr{G}) \cong \bigoplus_{p+q=n} H^p(X, \mathscr{F}) \otimes H^q(Y, \mathscr{G}).$$

The following result, which generalises [20, Lemma 9 parts (1) and (2)], characterises the image of L.

Proposition 4.5. *The linear map $L : \bigoplus_{r=1}^{M} \mathbb{C}^{e_r} \to \Gamma(\mathscr{B}(\boldsymbol{R}))$ in (4.3) is bijective.*

PROOF. By the definition of L, it suffices to show for each $e \in E$ that L_e is an injective linear map between vector spaces of the same dimension. First we show that L_e is injective. Consider $e \in E_r$ and let $T \in \mathbb{C}^{e_r}$. If $T \neq 0$, then there exist $\boldsymbol{x}_{s_j(e)} \in \mathbb{C}^{d_{s_j(e)}}$ for $j \in [\mu(e)]$ with $\boldsymbol{v} := T(\boldsymbol{x}_{s(e)}) \neq 0$. Let $\boldsymbol{x}_{t(e)} \in \mathbb{C}^{d_{t(e)}} \setminus \operatorname{span}(\boldsymbol{v})$. Then $L_e(T)(\chi)(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}) \neq \bar{0}$. Hence, the global section $L_e(T)$ is not the zero section.

We recursively apply the Künneth formula in the case $n = 0$ to obtain

$$H^0(X, \mathscr{B}(e)) = \bigotimes_{j=1}^{\mu(e)} H^0(X, \pi^*_{s_j(e)} \mathscr{H}(d_{s_j(e)})) \otimes H^0(X, \pi^*_{t(e)} \mathscr{Q}(d_{t(e)})).$$

It remains to compute the dimensions of the factors. We have $\dim H^0(X, \pi_i^* \mathscr{H}(d_i)) = d_i$ by results on the cohomology of line bundles over projective space [26, Theorem 5.1]. Finally, the short exact sequence (4.2) gives a long exact sequence in cohomology

$$0 \to \underbrace{H^0(X, \pi_i^* \mathscr{T}(d_i))}_{=0} \to H^0(X, \pi_i^* \mathscr{F}(d_i)) \to H^0(X, \pi_i^* \mathscr{Q}(d_i)) \to \underbrace{H^1(X, \pi_i^* \mathscr{T}(d_i))}_{=0} \to \cdots.$$

The underlined terms are 0, again by [26, Theorem 5.1]. Thus $\dim H^0(X, \pi_i^* \mathscr{Q}(d_i)) = d_i$, since $\dim H^0(X, \pi_i^* \mathscr{F}(d_i)) = d_i$. Hence $\dim H^0(X, \mathscr{B}(e)) = \prod_{j=1}^m d_{s_j(e)}$. This is the dimension of $\mathbb{C}^{e_r}$, so L_e is a bijection. $\square$

5. Bertini-type theorem

In this section, we relate the zeros of a generic section of a vector bundle to its top Chern class, cf. [20, Section 2.5]. This relation holds when the vector bundle is "almost generated", see Definition 5.2. We refer the reader to Appendix A for relevant background on Chern classes and Chow rings. In this section, X is any smooth complex projective variety. Recall that the global sections of $\mathscr{B}$, denoted $\Gamma(\mathscr{B})$, are the holomorphic maps $\sigma : X \to \mathscr{B}$ that send each $\chi \in X$ to a point in the fibre $\mathscr{B}_\chi$.

DEFINITION 5.1. Let X be a smooth projective variety and $\mathscr{B}$ a vector bundle over X. The vector bundle $\mathscr{B}$ is *globally generated* if there exists a vector subspace $\Lambda \subseteq \Gamma(\mathscr{B})$ such that for all $\chi \in X$, we have $\Lambda(\chi) = \mathscr{B}_\chi$, where $\Lambda(\chi) := \{\sigma(\chi) \mid \sigma \in \Lambda\}$.

DEFINITION 5.2. Let X be a smooth projective variety and $\mathscr{B}$ a vector bundle over X. The vector bundle $\mathscr{B}$ is *almost generated* if there exists a vector subspace $\Lambda \subseteq \Gamma(\mathscr{B})$ such that either $\mathscr{B}$ is globally generated, or there are $k \geq 1$ smooth irreducible proper subvarieties $Y_1, \ldots, Y_k$ of X, with $Y_0 = X$, such that:

(i) For all $i \geq 0$, there is a vector bundle $\mathscr{B}_i$ over Y_i, and for any $j \geq 0$, if Y_i is a subvariety of Y_j, then $\mathscr{B}_i$ is a subbundle of $\mathscr{B}_j|_{Y_i}$
(ii) $\Lambda(\chi) \subseteq (\mathscr{B}_i)_\chi$ for all $\chi \in Y_i$ and $i \geq 0$
(iii) If $\alpha_i \subseteq [k]$ is the set of all $j \in [k]$ such that Y_j is a proper subvariety of Y_i, then $\Lambda(\chi) = (\mathscr{B}_i)_\chi$ for all $\chi \in Y_i \setminus \left(\cup_{j \in \alpha_i} Y_j\right)$.

Now we state our Bertini-type theorem; cf. [20, Theorem 6]. The zero locus of a section $\sigma \in \Gamma(\mathscr{B})$ is $Z(\sigma) := \{\chi \in X \mid \sigma(\chi) = 0\}$. The top Chern class and top Chern number of

$\mathscr{B}$, see Definition A.5, are denoted $c_r(\mathscr{B}) \in A^*(X)$ and $\nu(\mathscr{B}) \in \mathbb{Z}$, respectively. We assume $X \subseteq \mathbb{P}^D$ via some closed immersion $s : X \hookrightarrow \mathbb{P}^D$ and regard $c_r(\mathscr{B}) = s_*(c_r(\mathscr{B})) \in A^*(\mathbb{P}^D)$, see Remark A.3.

THEOREM 5.3 (Bertini-Type Theorem). *Let $X \subseteq \mathbb{P}^D$ be a smooth irreducible complex projective variety of dimension d, and $\mathscr{B}$ a vector bundle of rank r over X, almost generated by a vector subspace $\Lambda \subseteq \Gamma(\mathscr{B})$. Let $\sigma \in \Lambda$ be a generic section with $Z(\sigma) \subseteq X$ its zero locus.*

(a) If $r > d$, then $Z(\sigma)$ is empty
(b) If $r = d$, then $Z(\sigma)$ consists of $\nu(\mathscr{B})$ points. Furthermore, if $\operatorname{rank} \mathscr{B}_i > \dim Y_i$ for all $i \geq 1$, then each point has multiplicity 1 and does not lie on $\cup_{i=1}^k Y_i$.
(c) If $r < d$, then $Z(\sigma)$ is empty or smooth of pure dimension $d - r$. In the latter case, the degree of $Z(\sigma)$ is $\nu\left(\mathscr{B}|_L\right)$, where $L \subseteq \mathbb{P}^D$ is the intersection of $d - r$ generic hyperplanes in $\mathbb{P}^D$. If $\nu\left(\mathscr{B}|_L\right) \neq 0$, then $Z(\sigma)$ is non-empty.

REMARK 5.4. The above theorem generalises [20, Theorem 6], where parts (a) and (b) appear. We add part (c). Compared to [20, Theorem 6], our extra assumption $\dim(\mathscr{B}_i) > \dim(Y_i)$ for $i > 0$ in (b) appears because it is absent from Definition 5.2, whereas it appears in [20, Definition 5].

To prove Theorem 5.3, we use the following results.

THEOREM 5.5 (Fiber Dimension Theorem [28, Theorem 1.25]). *Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Then there exists an open set $U \subseteq Y$ such that for all $y \in U$, $\dim X = \dim Y + \dim(f^{-1}(y))$.*

THEOREM 5.6 (Generic Smoothness Theorem [26, Corollary III.10.7]). *Let $f : X \to Y$ be a morphism of irreducible complex varieties. If X is smooth, then there exists an open subset $U \subseteq Y$ such that $f|_{f^{-1}(U)}$ is smooth. Furthermore, if f is not dominant, then $f^{-1}(U) = \varnothing$.*

PROOF OF THEOREM 5.3. Consider $I = \{(\chi, \sigma) \in X \times \Lambda \mid \sigma(\chi) = 0\}$ with projection maps

$$\begin{tikzcd} & I \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & \Lambda \end{tikzcd}$$

Then I is a vector bundle over X. Since the base space X is irreducible, so is the total space I. We show that $\dim I = \dim \Lambda + d - r$. The map p is surjective, and hence dominant, since the zero section lies in Λ. There exists an open set $U \subseteq X$ such that $\dim I = d + \dim(p^{-1}(\chi))$ for all $\chi \in U$, by Theorem 5.5. The fibre $p^{-1}(\chi) \simeq \{\sigma \in \Lambda : \sigma(\chi) = 0\}$ consists of sections in Λ that vanish at χ. Consider the evaluation map $\{\chi\} \times \Lambda \to \mathscr{B}_\chi$ that sends (χ, σ) to $\sigma(\chi)$. This is a linear map of vector spaces and its kernel is isomorphic to $p^{-1}(\chi)$. Let $Y := \cup_{i=1}^k Y_i$, where the Y_i are from Definition 5.2. The variety Y is a proper subvariety of X. For each $\chi \in X \setminus Y$, the evaluation map is surjective, by Definition 5.2(iii). Thus, the evaluation map has rank r and nullity $\dim \Lambda - r$. Hence $\dim(p^{-1}(\chi)) = \dim \Lambda - r$ for all $\chi \in U \cap (X \setminus Y)$. Therefore $\dim I = \dim \Lambda + d - r$.

The fiber $q^{-1}(\sigma) \simeq \{\chi \in X : \sigma(\chi) = 0\}$ is the zero locus $Z(\sigma)$. We show that the map q is dominant if and only if $q^{-1}(\sigma) \neq \varnothing$ for generic $\sigma \in \Lambda$. If q is dominant, then there exists an open set $W \subseteq \Lambda$ such that $q^{-1}(\sigma)$ is smooth of codimension $\dim I - \dim \Lambda = d - r$ for all $\sigma \in W$, by Theorems 5.5 and 5.6. In particular, $q^{-1}(\sigma)$ is non-empty. Conversely if q is not dominant, then there is an open set $W \subseteq \Lambda$ such that $q^{-1}(\sigma) = \varnothing$ for all $\sigma \in W$, by Theorem 5.6.

Now we show that $Z(\sigma) \neq \varnothing$ for generic $\sigma \in \Lambda$ if and only if $c_r(\mathscr{B}) \neq 0$. If $Z(\sigma) = \varnothing$, then the existence of a nowhere vanishing section of $\mathscr{B}$ implies that $c_r(\mathscr{B}) = 0$ [21, Lemma 3.2]. Conversely, if $Z(\sigma) \neq \varnothing$, then the map q is dominant, so $Z(\sigma)$ is smooth of codimension $d - r$. If $c_r(\mathscr{B}) = 0$, then $0 = c_r(\mathscr{B}) = [Z(\sigma)]$ by Definition A.5(ii), which is a contradiction since the degree of a non-empty projective variety is a positive integer [26, Proposition I.7.6.a]. In particular, if $r = d$ and $\nu(\mathscr{B}) = 0$, then $Z(\sigma) = \varnothing$.

The map q is not dominant if $\dim I < \dim \Lambda$; i.e., if $r > d$. This proves (a) and the emptiness possibility in (c). It remains to consider the case $r \leq d$ with the map q dominant and generic $\sigma \in \Lambda$.

$Z(\sigma) \subseteq \mathbb{P}^D$ is smooth of dimension $d - r$. It is pure dimensional by [21, Example 3.2.16]. When $r = d$, we have $[Z(\sigma)] = c_r(\mathscr{B}) = \nu(\mathscr{B})[p]$ for some $p \in X$, by Definition A.5(ii), so the zero locus consists of $\nu(\mathscr{B})$ points. It remains to relate the degree to the top Chern class for $r < d$. The degree of $Z(\sigma)$ is the number of points in the intersection of $Z(\sigma)$ with $d - r$ generic hyperplanes $\mathbb{P}^D$. Denote the intersection of $d - r$ such hyperplanes by L. Let $L \xhookrightarrow{j} \mathbb{P}^D$ be its inclusion. We have $[Z(\sigma)] = c_r(\mathscr{B})$ by Definition A.5(ii) and seek $[L]c_r(\mathscr{B})$. We compute in $A^*(\mathbb{P}^D)$:

$$
\begin{aligned}
[L]c_r(\mathscr{B}) &= j_*([L])c_r(\mathscr{B}) && \text{(definition of pushforward)} \\
&= j_*(j^*(c_r(\mathscr{B}))[L]) && \text{(projection formula)} \\
&= j_*(c_r(j^*\mathscr{B})[L]) = j_*(c_r\left(\mathscr{B}\big|_L\right)[L]) && \text{(Definition A.5(iv))} \\
&= j_*(\nu\left(\mathscr{B}\big|_L\right)[p][L]) && \text{(definition of top Chern number)} \\
&= \nu\left(\mathscr{B}\big|_L\right) j_*([p][L]) && \text{(pushforward is a morphism)} \\
&= \nu\left(\mathscr{B}\big|_L\right) j_*([p]) = \nu\left(\mathscr{B}\big|_L\right)[p] && \text{(intersection with a point)} \qquad (5.1)
\end{aligned}
$$

for some point $p \in L$. Thus, the degree of $Z(\sigma)$ is $\nu\left(\mathscr{B}\big|_L\right)$. As a corollary, we obtain that if $\nu(\mathscr{B}) \neq 0$ or $\nu\left(\mathscr{B}\big|_L\right) \neq 0$, then $Z(\sigma) \neq \varnothing$. This proves the dimension and degree statements in (b) and (c).

Lastly, we show that when $r = d$ and the additional assumptions of (b) hold, the points in $Z(\sigma)$ are generically of multiplicity 1 and do not lie on Y. Smoothness in Theorem 5.6 shows that each of the finitely many points in $q^{-1}(\sigma)$ are of multiplicity 1. We have $\operatorname{rank} \mathscr{B}_i > \dim Y_i$ for all $i \geq 1$. Hence $\dim(p^{-1}(Y_i)) = \dim Y_i + \dim \Lambda - \operatorname{rank} \mathscr{B}_i < \dim \Lambda$. Thus, $\dim(p^{-1}(Y)) < \dim \Lambda$, and using the fact that the projection $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is a closed map, we deduce that q is a closed map. Hence $q(p^{-1}(Y))$ is a proper subvariety of Λ. For all points in the open set $\sigma \in W \cap W'$, where $W' = \Lambda \setminus q(p^{-1}(Y))$, the fibre $q^{-1}(\sigma)$ contains no points in Y. $\square$

REMARK 5.7. Our proof of Theorem 5.3, is analogous to the proofs in [20] of their Theorems 4 and 6. Their proof uses [21, Example 3.2.16], which is equivalent to axiom (ii) in Definition A.5. Our proof adds the Chern number computation for case (c).

6. Generating the singular vector bundle

In this section we show that $\mathscr{B}(\boldsymbol{R})$ is almost generated, so that Theorem 5.3 may be applied to it. We generalise the singular vector bundle to a bundle $\mathscr{B}(\boldsymbol{R}, F)$, for a subset of hyperedges $F \subseteq E$. The zeros of a global section of $\mathscr{B}(\boldsymbol{R}, F)$ are singular vectors with singular value zero along the edges in F. We show that $\mathscr{B}(\boldsymbol{R}, F)$ is almost generated. This will later yield not only the dimension and degree of the singular vector variety $\mathcal{S}(\boldsymbol{R})$ in Theorem 3.1, but also the final statement about the non-existence of a zero singular value.

DEFINITION 6.1. Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a hyperquiver representation and let $X = \prod_{i=1}^n \mathbb{P}(\mathbb{C}^{d_i})$. Given $F \subseteq E$, we define

$$\mathscr{B}(e, F) = \begin{cases} \operatorname{Hom}\left(\mathscr{T}(e), \pi_{t(e)}^* \mathscr{Q}(d_{t(e)})\right) & \text{if } e \notin F \\ \operatorname{Hom}\left(\mathscr{T}(e), \pi_{t(e)}^* \mathscr{F}(d_{t(e)})\right) & \text{if } e \in F. \end{cases}$$

It has fibres

$$\mathscr{B}(e, F)_\chi = \begin{cases} \operatorname{Hom}\left(\operatorname{span}\left(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}\right), \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)})\right) & \text{if } e \notin F \\ \operatorname{Hom}\left(\operatorname{span}\left(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}\right), \mathbb{C}^{d_{t(e)}}\right) & \text{if } e \in F, \end{cases}$$

where $\chi = ([\boldsymbol{x}_1], \ldots, [\boldsymbol{x}_n])$. The *singular vector bundle* of $\boldsymbol{R}$ over X with respect to F is $\mathscr{B}(\boldsymbol{R}, F) = \bigoplus_{e \in E} \mathscr{B}(e, F)$.

The singular vector bundle $\mathscr{B}(\boldsymbol{R})$ from Definition 4.1 is $\mathscr{B}(\boldsymbol{R}, \varnothing)$.

PROPOSITION 6.2. *The bundle $\mathscr{B}(\boldsymbol{R}, F)$ has rank $\sum_{e \in E}(d_{t(e)} - 1) + |F|$.*

PROOF. The rank of $\mathscr{B}(\boldsymbol{R}, F)$ is $\sum_{e \in E} \operatorname{rank} \mathscr{B}(e, F)$. For $e \notin F$, $\operatorname{rank} \mathscr{B}(e, F) = d_{t(e)} - 1$, as in Proposition 4.2(b). For $e \in F$, $\operatorname{rank} \mathscr{B}(e, F) = \operatorname{rank} \operatorname{Hom}(\mathscr{T}(e), \pi_{t(e)}^* \mathscr{F}(d_{t(e)})) = d_{t(e)}$. $\square$

We construct global sections for $\mathscr{B}(\boldsymbol{R}, F)$ whose zero loci correspond to singular vectors with zero singular value along the edges in F. Define the map

$$\begin{aligned} L_{e,F} : \mathbb{C}^{e_r} &\longrightarrow \Gamma(\mathscr{B}(e, F)) \\ L_{e,F}(T)(\chi)(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)}) &= \begin{cases} \overline{T(\boldsymbol{x}_{s(e)})} \in \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)}) & e \notin F \\ T(\boldsymbol{x}_{s(e)}) \in \mathbb{C}^{d_{t(e)}} & e \in F. \end{cases} \end{aligned} \tag{6.1}$$

We define the composite map

$$L_F : \bigoplus_{r=1}^{M} \mathbb{C}^{e_r} \to \Gamma(\mathscr{B}(\boldsymbol{R}, F)) \tag{6.2}$$

$$L_F = \bigoplus_{r=1}^{M} \bigoplus_{e \in E_r} L_{e,F}.$$

We connect the global sections in the image of L_F to the singular vector tuples of $\boldsymbol{R}$, generalizing Proposition 4.3 and [20, Lemma 11].

PROPOSITION 6.3. *Let $\mathscr{B}(\boldsymbol{R}, F)$ be the singular vector bundle with respect to F and $L_F : \bigoplus_{r=1}^{M} \mathbb{C}^{e_r} \to \Gamma(\mathscr{B}(\boldsymbol{R}, F))$ the linear map in (6.2). A point $\boldsymbol{\chi} = ([\boldsymbol{x}_1], \ldots, [\boldsymbol{x}_n]) \in X$ lies in the zero locus of the section $\sigma = L_F((T_r)_{r=1}^{M})$ if and only if $\boldsymbol{\chi}$ is a singular vector tuple of $\boldsymbol{R}$ with zero singular value along all edges in F.*

PROOF. The image $L_F((T_r)_{r=1}^{M})(\boldsymbol{\chi})$ is the tuple of zero maps each in $\mathscr{B}(e, F)_{\boldsymbol{\chi}}$ if and only if for all $e \in E_r$ and $r \in [M]$, $L_{e,F}(T_r)(\boldsymbol{\chi})(\otimes_{j=1}^{\mu(e)} \boldsymbol{x}_{s_j(e)})$ is the zero vector in the appropriate case of (6.1), if and only if $T_r(\boldsymbol{x}_{s(e)}) = \lambda_e \boldsymbol{x}_{t(e)}$ for some $\lambda_e \in \mathbb{C}$ with $\lambda_e = 0$ if $e \in F$, if and only if $\boldsymbol{\chi}$ is a singular vector tuple of the hyperquiver representation $\boldsymbol{R}$, with zero singular values along the edges of F. $\square$

DEFINITION 6.4. The *isotropic quadric* $Q_n = \{\boldsymbol{v} \in \mathbb{C}^n : \boldsymbol{v}^\top \boldsymbol{v} = 0\}$ is the quadric hypersurface in $\mathbb{C}^n$ of isotropic vectors. The variety Q_n is defined by a homogeneous equation. We consider it as a subvariety $\mathbb{P}(Q_n)$ of $\mathbb{P}^n$.

DEFINITION 6.5. If $T \in \mathbb{C}^e$ is a tensor and $\boldsymbol{x}_{s_j(e)} \in \mathbb{C}^{d_{s_j(e)}}$ are vectors for $j \in [m]$, then we denote by $T(\boldsymbol{x}_e) := T_e(\boldsymbol{x}_{t(e)}, \boldsymbol{x}_{s_1(e)}, \ldots, \boldsymbol{x}_{s_\mu(e)}) = \boldsymbol{x}_{t(e)}^\top T(\boldsymbol{x}_{s(e)}) \in \mathbb{C}$ the contraction of the tensor T by the vectors $\boldsymbol{x}_{s_j(e)}$, where $T(\boldsymbol{x}_{s(e)})$ is the vector defined in (2.1).

We give a necessary and sufficient condition for when the maps in (6.1) generate the vector space $\mathscr{B}(e)_{\boldsymbol{\chi}}$. This generalises [20, Lemma 8] from a single tensor to a hyperquiver representation. Later, in our proof that $\mathscr{B}(\boldsymbol{R}, F)$ is almost generated, we apply this condition to the vector subbundles $\mathscr{B}_i$ in Definition 5.2. This will allow us to associate a single tensor to each piece of the partition.

LEMMA 6.6. *Let $H = (V, E)$ be a hyperquiver, $E = \coprod_{r=1}^{M} E_r$ be a partition, and assign vector spaces $\mathbb{C}^{d_i}$ to each vertex $i \in V$. Fix a collection of vectors $\boldsymbol{x}_i \in \mathbb{C}^{d_i} \setminus \{0\}$ for $i \in [n]$ and $\boldsymbol{y}_e \in \mathbb{C}^{d_{t(e)}}$ for $e \in E$. Fix $F \subseteq E$ a subset of hyperedges. Let G_r be the hyperedges $e \in E_r \setminus F$ such that $\boldsymbol{x}_{t(e)}$ is isotropic. Then for all $r \in [M]$, the following are equivalent:*

(a) There exist tensors $T_r \in \mathbb{C}^{e_r}$ for some $e_r \in E_r$ satisfying the equations

$$\overline{T_r(\boldsymbol{x}_{s(e)})} = \overline{\boldsymbol{y}_e} \in \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)}) \qquad e \in E_r \setminus F \tag{6.3}$$

$$T_r(\boldsymbol{x}_{s(e)}) = \boldsymbol{y}_e \in \mathbb{C}^{d_{t(e)}} \qquad e \in E_r \cap F. \tag{6.4}$$

(b) Given any pair of edges $e, e' \in (F \cap E_r) \cup G_r$, we have

$$\boldsymbol{x}_{t(e)}^\top \boldsymbol{y}_e = \boldsymbol{x}_{t(e')}^\top \boldsymbol{y}_{e'}. \tag{6.5}$$

PROOF. $(a \Rightarrow b)$: There is a tensor T_r satisfying (6.3) if and only if there are scalars $\lambda_e \in \mathbb{C}$ such that $T_r(\boldsymbol{x}_{s(e)}) = \boldsymbol{y}_e + \lambda_e \boldsymbol{x}_{t(e)}$ for all $e \in E_r \setminus F$. Multiplying both sides by $\boldsymbol{x}_{t(e)}$ gives $T_r(\boldsymbol{x}_e) = \boldsymbol{x}_{t(e)}^\top \boldsymbol{y}_e + \lambda_e \boldsymbol{x}_{t(e)}^\top \boldsymbol{x}_{t(e)}$. Similarly, from (6.4) we obtain, for $e \in F \cap E_r$, the condition $T_r(\boldsymbol{x}_e) = \boldsymbol{x}_{t(e)}^\top \boldsymbol{y}_e$. The scalar $T_r(\boldsymbol{x}_e)$ only depends on r via the set E_r. Thus for any pair of edges $e, e' \in E_r$, we have

$$\boldsymbol{x}_{t(e)}^\top \boldsymbol{y}_e + \lambda_e \boldsymbol{x}_{t(e)}^\top \boldsymbol{x}_{t(e)} = \boldsymbol{x}_{t(e')}^\top \boldsymbol{y}_{e'} + \lambda_{e'} \boldsymbol{x}_{t(e')}^\top \boldsymbol{x}_{t(e')}$$

where $\lambda_e = 0$ for $e \in F \cap E_r$. For the hyperedges in G_r, the terms $\boldsymbol{x}_{t(e)}^\top \boldsymbol{x}_{t(e)}$ vanish. Hence (6.5) holds for all $e, e' \in (F \cap E_r) \cup G_r$.

$(b \Rightarrow a)$: Let $\mu_r \in \mathbb{C}$ be the value of (6.5) if $(F \cap E_r) \cup G_r \neq \varnothing$ and zero otherwise. Define

$$\lambda_e = \begin{cases} 0 & e \in (F \cap E_r) \cup G_r \\ (\boldsymbol{x}_{t(e)}^\top \boldsymbol{x}_{t(e)})^{-1} (\mu_r - \boldsymbol{x}_{t(e)}^\top \boldsymbol{y}_e) & \text{otherwise.} \end{cases}$$

Choose some $e_r \in E_r$. We show that, for such a choice of λ_e, there exists a tensor $T_r \in \mathbb{C}^{e_r}$ that satisfies

$$T_r(\boldsymbol{x}_{s(e)}) = \boldsymbol{y}_e + \lambda_e \boldsymbol{x}_{t(e)} \tag{6.6}$$

for all $e \in E_r$, and hence there exists a tensor T_r that satisfies (6.3) and (6.4). A change of basis in each $\mathbb{C}^{d_i}$ does not affect the existence or non-existence of solutions to (6.6). Consider the change of basis that sends each $\boldsymbol{x}_i$ to the first standard basis vector in $\mathbb{C}^{d_i}$, which we denote by $\boldsymbol{e}_{i,1} = (1, 0, \dots, 0)^\top$. For each $e \in E_r$, there is a permutation σ of $[m]$ sending $v(e)$ to $v(e_r)$ by Definition 2.6(i.b). Then (6.6) becomes the condition

$$(T_r)_{1,\dots,1,\ell,1,\dots,1} = (\boldsymbol{y}_e)_\ell + \lambda_e \delta_{1,\ell} \text{ for all } \ell \in [d_{t(e)}],$$

where $\delta_{i,j}$ is the Kronecker delta and the ℓ on the left hand side appears in position $\sigma(m)$. We define T_r to be the tensor whose non-zero entries are given by the above equation. This is well-defined, since $\sigma(m) \neq \sigma'(m)$ for $\sigma \neq \sigma'$, by Definition 2.6(i.c). It remains to show that we do not attempt to assign different values to the same entry of T_r. When $\ell = 1$, we assign the value $(\boldsymbol{y}_e)_1 + \lambda_e$. For all edges this quantity equals $(\boldsymbol{y}_e)_1 = \mu_r$. □

To conclude this section, we show that $\mathscr{B} := \mathscr{B}(\boldsymbol{R}, F)$ satisfies the conditions of Definition 5.2. This shows that $\mathscr{B}$ is almost generated. First we define the subvarieties Y_i and the vector bundles $\mathscr{B}_i$ over Y_i that appear in Definition 5.2.

We use the following notation. A linear functional $\varphi : \mathbb{C}^{d_{t(e)}} / \operatorname{span}(\boldsymbol{x}_{t(e)}) \to \mathbb{C}$ can be uniquely represented by a vector $\boldsymbol{u} \in \mathbb{C}^{d_{t(e)}}$ such that $\boldsymbol{u}^\top \boldsymbol{x}_{t(e)} = 0$ and $\varphi([\boldsymbol{y}]) = \boldsymbol{u}^\top \boldsymbol{y}$, [20, Lemma 7]. In particular when $\boldsymbol{x}_{t(e)} \in Q_{t(e)}$, we abbreviate $\boldsymbol{x}_{t(e)}^\top [\boldsymbol{y}]$ to $\boldsymbol{x}_{t(e)}^\top \boldsymbol{y}$.

For a subset $\alpha \subseteq [n]$, define the smooth proper irreducible subvariety

$$Y_\alpha = X_1 \times \cdots \times X_n, \quad \text{where} \quad X_i = \begin{cases} \mathbb{P}(Q_i) & i \in \alpha \\ \mathbb{P}(\mathbb{C}^{d_i}) & i \notin \alpha. \end{cases}$$

In particular, $Y_\varnothing = X$. Fix $F \subseteq E$ and define $F' = \{t(e)\}_{e \in F}$. Fix $\alpha \subseteq [n] \setminus F'$. Let $G_r \subseteq E_r \setminus F$ denote the edges whose target vertex lies in α. Define $\mathscr{B}_\alpha$ to be the vector bundle over Y_α

whose fiber at $\chi = ([x_1], \ldots, [x_n]) \in Y_\alpha$ is the subspace $U(\alpha, \chi)$ of linear maps $\tau = (\tau_e)_{e \in E} \in (\mathscr{B})_\chi$ satisfying

$$x_{t(e)}^\top \tau_e(\otimes_{j=1}^{\mu(e)} x_{s_j(e)}) = x_{t(e')}^\top \tau_{e'}(\otimes_{j=1}^{\mu(e')} x_{s_j(e')}), \tag{6.7}$$

for any edges $e, e' \in (F \cap E_r) \cup G_r$, for every $r \in [M]$.

PROPOSITION 6.7. *Let the map L_F be as in (6.2). For any subset of hyperedges $F \subseteq E$, the vector subspace $L_F\left(\bigoplus_{r=1}^M \mathbb{C}^{e_r}\right)$ almost generates $\mathscr{B}(\boldsymbol{R}, F)$.*

PROOF. We first show that the vector bundles $\mathscr{B}_\alpha$ satisfy Definition 5.2(i). If $\alpha, \beta \subseteq [n] \setminus F'$, then $\alpha \subsetneq \beta$ if and only if Y_β is a proper subvariety of Y_α. Furthermore, $\mathscr{B}_\beta$ is a subbundle of $\mathscr{B}_\alpha|_{Y_\alpha}$, since $U(\beta, \chi)$ is a vector subspace of $U(\alpha, \chi)$.

Next we prove that Definition 5.2(ii) holds. Recall that $\Lambda(\chi) := \{\sigma(\chi) \mid \sigma \in \Lambda\}$. We show that $\Lambda(\chi) \subseteq (\mathscr{B}_\alpha)_\chi$. If $\chi \in Y_\alpha$, then an element of $\Lambda(\chi)$ is an $|E|$-tuple of linear maps $L_{e,F}(T_r)(\chi)$ for some tensors $T_r \in \mathbb{C}^{e_r}$, $r \in [M]$. By the proof of $(a \Rightarrow b)$ in Lemma 6.6, $\tau_e := L_{e,F}(T_r)(\chi)$ satisfy (6.7), so $\Lambda(\chi) \subseteq (\mathscr{B}_\alpha)_\chi$.

Finally we show that Definition 5.2(iii) holds. If χ lies on Y_α but not on any proper subvariety Y_β, then every $(\tau_e)_{e \in E} \in (\mathscr{B}_\alpha)_\chi$ satisfies (6.7) and no additional equations. Thus there exist tensors T_r with $L_{e,F}(T_r) = \tau_e$ for $e \in E_r$ and $\tau \in \Lambda(\chi)$, by Lemma 6.6. Hence, $\Lambda(\chi) = (\mathscr{B}_\alpha)_\chi$. □

7. The top Chern class of the singular vector bundle

In this section we compute the top Chern class of the singular vector bundle $\mathscr{B}(\boldsymbol{R})$, generalizing [20, Lemma 3]. Combining this computation with Theorem 5.3 and Proposition 6.7 finds the degree of the singular vector variety, completing the proof of Theorem 3.1.

PROPOSITION 7.1. *Let $\boldsymbol{R} = (\boldsymbol{d}, T)$ be a hyperquiver representation and $\mathscr{B}(\boldsymbol{R})$ be the singular vector bundle over $X = \prod_{i=1}^n \mathbb{P}(\mathbb{C}^{d_i})$. Then the top Chern class of $\mathscr{B}(\boldsymbol{R})$ is*

$$\prod_{e \in E} \sum_{k=1}^{d_{t(e)}} h_{t(e)}^{k-1} h_{s(e)}^{d_{t(e)}-k}, \quad \text{where} \quad h_{s(e)} = \sum_{j=1}^{\mu(e)} h_{s_j(e)},$$

in the Chow ring $A^*(X) \cong \mathbb{Z}[h_1, \ldots, h_n]/(h_1^{d_1}, \ldots, h_n^{d_n})$.

PROOF. We seek the Chern polynomial $C(t, \mathscr{B}(\boldsymbol{R}))$. The coefficient of its highest power of t is the top Chern class. The Chern polynomial is multiplicative over short exact sequences, see Definition A.5(iii). Hence

$$C(t, \mathscr{F}(d)) = C(t, \mathscr{T}(d))C(t, \mathscr{Q}(d)), \tag{7.1}$$

by (4.1). We compute $C(t, \mathscr{T}(d))$. Let $h \in A^*(\mathbb{P}(\mathbb{C}^d)) \cong \mathbb{Z}[h]/(h^d)$ be the class of a hyperplane in $\mathbb{P}(\mathbb{C}^d)$. By Definition A.5(i)-(ii), h is the first Chern class $c_1(\mathscr{H}(d))$ and the Chern polynomial of $\mathscr{H}(d)$ is $C(t, \mathscr{H}(d)) = 1 + ht$. Thus $C(t, \mathscr{T}(d)) = C(-t, \mathscr{H}(d)^\vee) = 1 - ht$, by Proposition A.8(b).

Next we compute $C(t, \mathscr{Q}(d))$. We have $C(t, \mathscr{F}(d)) = 1$, by Proposition A.8(a). The Chern polynomial of $\mathscr{Q}(d)$ is the inverse of $(1 - ht)$, by (7.1). Using the formal factorization $1 -$

$x^n = \prod_{k=0}^{n}(1-\zeta_n^k x)$ over $A^*(X)\otimes\mathbb{C}$, we therefore have

$$C(t,\mathscr{Q}(d)) = \sum_{j=0}^{d-1}(ht)^j = \frac{1-(ht)^{d-1}}{1-ht} = \frac{\prod_{k=0}^{d-1}(1-\zeta_d^k ht)}{1-ht} = \prod_{k=1}^{d-1}(1-\zeta_d^k ht)$$

where $\zeta_d \in \mathbb{C}$ is a d-th root of unity.

We have $c_1(\pi_i^*\mathscr{H}(d_i)) = \pi_i^* c_1(\mathscr{H}(d_i)) = \pi_i^* h_i = h_i \in A^*(X)$, by Definition A.5(iv) and Definition A.2(ii). Thus the Chern polynomials of $\pi_i^*\mathscr{H}(d_i)$, $\pi_i^*\mathscr{T}(d_i)$, and $\pi_i^*\mathscr{Q}(d_i)$ equal those of $\mathscr{H}(d)$, $\mathscr{T}(d)$, and $\mathscr{Q}(d)$ respectively but with h replaced by $h_i \in A^*(X)$, by (4.2).

We have found the Chern roots of $\pi_i^*\mathscr{H}(d_i)$ and $\pi_i^*\mathscr{Q}(d_i)$, so we obtain Chern characters $\mathrm{ch}(\pi_i^*\mathscr{H}(d_i)) = \exp(h_i)$ and $\mathrm{ch}(\pi_i^*\mathscr{Q}(d_i)) = \sum_{k=1}^{d_i-1}\exp(-\zeta_{d_i}^k h_i)$. By Propositions 4.2(c) and A.8(c), the Chern character $\mathrm{ch}(\mathscr{B}(e))$ equals

$$\begin{aligned}
\mathrm{ch}\left(\bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^*\mathscr{H}(d_{s_j(e)}) \otimes \pi_{t(e)}^*\mathscr{Q}(d_{t(e)})\right) &= \mathrm{ch}\left(\bigotimes_{j=1}^{\mu(e)} \pi_{s_j(e)}^*\mathscr{H}(d_{s_j(e)})\right)\mathrm{ch}(\pi_{t(e)}^*\mathscr{Q}(d_{t(e)})) \\
&= \left(\prod_{j=1}^{\mu(e)}\exp(h_{s_j(e)})\right)\left(\sum_{k=1}^{d_{t(e)}-1}\exp(-\zeta_{d_{t(e)}} h_{t(e)})\right) \\
&= \sum_{k=1}^{d_{t(e)}-1}\exp\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)} - \zeta_{d_{t(e)}}^k h_{t(e)}\right).
\end{aligned}$$

Switching to Chern polynomial form, we obtain

$$C(t,\mathscr{B}(e)) = \prod_{k=1}^{d_{t(e)}-1}\left(1+\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)} - \zeta_{d_{t(e)}}^k h_{t(e)}\right)t\right).$$

This product has degree $(d_{t(e)}-1)$ in t, with top coefficient

$$\prod_{k=1}^{d_{t(e)}-1}\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)} - \zeta_{t(e)}^k h_{t(e)}\right).$$

It follows from Definition A.5(iii) that $C(t,\mathscr{B}(\boldsymbol{R})) = \prod_{e\in E} C(t,\mathscr{B}(e))$. The product has degree $(\sum_{e\in E} d_{t(e)} - |E|)$ in t, with top coefficient (i.e., top Chern class of $\mathscr{B}(\boldsymbol{R})$) equal to

$$\prod_{e\in E}\prod_{k=1}^{d_{t(e)}-1}\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)} - \zeta_{t(e)}^k h_{t(e)}\right).$$

Finally, the formal identity $x^n - y^n = \prod_{k=0}^{n}(x-\zeta_n^k y)$ gives

$$\begin{aligned}
\prod_{e\in E}\prod_{k=1}^{d_{t(e)}-1}\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)} - \zeta_{t(e)}^k h_{t(e)}\right) &= \prod_{e\in E}\frac{\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)}\right)^{d_{t(e)}-1} - h_{t(e)}^{d_{t(e)}-1}}{\sum_{j=1}^{\mu(e)} h_{s_j(e)} - h_{t(e)}} \\
&= \prod_{e\in E}\sum_{k=0}^{d_{t(e)}-1}\left(\sum_{j=1}^{\mu(e)} h_{s_j(e)}\right)^{d_{t(e)}-1-k} h_{t(e)}^k \in A^*(M). \qquad \square
\end{aligned}$$

To conclude the paper, we now prove our main theorem.

PROOF OF THEOREM 3.1. The zero locus of a generic global section of $\mathscr{B} := \mathscr{B}(\boldsymbol{R}, F)$ is the singular vector variety $\mathcal{S}(\boldsymbol{R})$, with zero singular values along the edges in F, by Propositions 4.3 and 6.3. The singular vector bundle $\mathscr{B}$ from Definition 6.1 is almost generated, by Proposition 6.7. Hence our Bertini-type theorem Theorem 5.3 applies to it, to characterise the zeros of a generic section. It remains to derive the polynomial (3.1), prove the emptiness statement for $\mathcal{S}(\boldsymbol{R})$ as well as its dimension and degree, and prove the statement regarding finitely many singular vector tuples.

We first consider the case $F = \varnothing$. The top Chern class $c_r(\mathscr{B})$ is given by Proposition 7.1. If $N = d - r = 0$, then $\mathcal{S}(\boldsymbol{R})$ has the claimed number of points by Theorem 5.3(b). Suppose $r < d$. Let $s : X \hookrightarrow \mathbb{P}^D$ be the Segre embedding and let $[l] \in A^*(\mathbb{P}^D)$ be the class of a hyperplane. Continuing (5.1), we have

$$\begin{aligned} \nu\left(\mathscr{B}|_L\right)[p] &= [L]c_r(\mathscr{B}) = [L]s_*(c_r(\mathscr{B})) = [l]^N s_*(c_r(\mathscr{B})) && \text{(definition of pushforward)} \\ &= s_*(s^*([l]^N)c_r(\mathscr{B})) = s^*([l]^N)c_r(\mathscr{B}) && \text{(projection formula)} \\ &= s^*([l])^N c_r(\mathscr{B}) = (\textstyle\sum_{i=1}^n h_i)^N c_r(\mathscr{B}) && \text{([21, Example 8.4.3])} \end{aligned} \tag{7.2}$$

where $A^*(X) \cong \mathbb{Z}[h_1, \ldots, h_n]/(h_1^{d_1}, \ldots, h_n^{d_n})$, giving us the polynomial (3.1).

We prove the emptiness statement by showing that $\nu\left(\mathscr{B}|_L\right) = 0$ if and only if $c_r(\mathscr{B}) = 0$. By the proof of Theorem 5.3, $c_r(\mathscr{B}) = 0$ if and only if $\mathcal{S}(\boldsymbol{R}) = \varnothing$. If $c_r(\mathscr{B}) = 0$, then $\nu\left(\mathscr{B}|_L\right) = 0$ by (7.2). Conversely, if $c_r(\mathscr{B}) \neq 0$, then there exists a monomial $h_1^{a_1} \ldots h_n^{a_n}$ in $c_r(\mathscr{B})$ such that $a_i < d_i$ and $\sum_{i=1}^n a_i = r$. There exists a monomial $h_1^{d_1-1-a_1'} \ldots h_n^{d_n-1-a_n'}$ in $(\sum_{i=1}^n h_i)^{d-r}$ such that $\sum_{i=1}^n a_i' = r$. Thus, these monomials pair in the product $[L]c_r(\mathscr{B})$ to form the monomial $[p] = h_1^{d_1-1} \ldots h_n^{d_n-1}$. The coefficient of this monomial is $\nu\left(\mathscr{B}|_L\right)$, which is non-zero. Therefore if $\nu\left(\mathscr{B}|_L\right) \neq 0$, $\mathcal{S}(\boldsymbol{R})$ has the claimed dimension and degree by Theorem 5.3.

It remains to prove the last sentence of the theorem, which pertains to the case $N = 0$. Fix $\varnothing \neq \alpha \subseteq [n]$ and define $\mathscr{B}_\alpha$ as in the proof of Proposition 6.7. Then $\operatorname{rank} \mathscr{B}_\alpha = \operatorname{rank} \mathscr{B} - (|\alpha| - 1) > \operatorname{rank} \mathscr{B} - |\alpha| = \dim(X) - |\alpha| = \dim(Y_\alpha)$ as the fibers of $\mathscr{B}_\alpha$ are vector subspaces of the fibers of $\mathscr{B}$ cut down by $|\alpha| - 1$ linearly independent equations (6.7). Thus, every singular vector has multiplicity 1 and is non-isotropic by Theorem 5.3(b). Finally, if $F \neq \varnothing$ then $\operatorname{rank} \mathscr{B} > \dim(X)$ by (6.2), so $\boldsymbol{R}$ has no singular values equal to 0, by Theorem 5.3(a). □

Acknowledgements. We thank Giorgio Ottaviani and Shmuel Friedland for helpful discussions. AS was supported by the Society of Fellows at Harvard University. VN is supported by the EPSRC grant EP/R018472/1 and US AFOSR grant FA9550-22-1-0462.

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Appendix A. The Chow ring and Chern classes

We recall the definitions of the Chow groups and Chow ring of a projective variety, following [16, 21].

DEFINITION A.1. Let X be a smooth projective variety of dimension n.

(i) [16, Section 1.2.1] The group of *i-cycles* of X is the free abelian group $Z_i(X)$ generated by the irreducible i-dimensional subvarieties of X. An element of $Z_i(X)$, called an *i-cycle*, is a finite, formal sum $\sum_i n_i V_i$ of i-dimensional subvarieties V_i of X, where $n_i \in \mathbb{Z}$.

(ii) [21, Proposition 1.6] An i-cycle $Z \in Z_i(X)$ is *rationally equivalent to zero* if there exist irreducible subvarieties $V_i \subseteq \mathbb{P}^1 \times X$ of dimension $i+1$ with dominant projection maps $V_i \to \mathbb{P}^1$ such that $Z = \sum_i V_i(0) - V_i(\infty)$, where $V_i(t) = V_i \cap (\{t\} \times X)$. The i-cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_i(X)$ of $Z_i(X)$.

(iii) [21, Section 1.2.2-1.2.3] The *i-th Chow group* of X is the quotient group $A_i(X) = Z_i(X)/\mathrm{Rat}_i(X)$. The class of an i-cycle $C \in Z_i(X)$ in $A_i(X)$ is denoted by $[C]$. The *Chow group* of X is the direct sum $A_*(X) = \oplus_{i=0}^n A_i(X)$. The *Chow ring* of X is the direct sum $A^*(X) = \oplus_{i=0}^n A^i(X)$, where $A^i(X) = A_{n-i}(X)$.

The Chow ring $A^*(X)$ has the structure of a commutative ring, with a product $A^i(X) \times A^j(X) \to A^{i+j}(X)$ called the *intersection product*. We say that C and D *intersect transversely* if on each component of $C \cap D$ at a generic point p, the sum of the tangent spaces of C and D is the tangent space of X: $T_pC + T_pD = T_pX$. The intersection product takes any codimension-i and codimension-j irreducible subvarieties $C, D \subseteq X$, replaces C and D by rationally equivalent subvarieties $C', D' \subseteq X$ (if necessary) in order for C' and D' to intersect transversely, and defines $[C][D] = [C' \cap D'] \in A^{i+j}(X)$. The existence of a well-defined intersection product is due to Fulton [21]; see [16, Appendix A].

DEFINITION A.2 ([16, Section 1.3.6]). Let X and Y be smooth projective varieties of dimensions m and n, and $f : X \to Y$ a morphism.

(i) Let $V \subseteq X$ be an irreducible subvariety of dimension i. Define a group homomorphism $f_* : A_i(X) \to A_i(Y)$ by
$$[V] \mapsto \begin{cases} d \cdot [f(V)] & \dim f(V) = i \\ 0 & \dim f(V) < i, \end{cases}$$
where $d := [R(V) : R(f(V))]$ is the degree of the field extension between the function fields $R(V)$ of V and $R(f(V))$ of $f(V)$. The map f_* extends to a group homomorphism $f_* : A_*(X) \to A_*(Y)$, called the *pushforward* of f.

(ii) There is a unique group homomorphism $f^* : A^i(Y) \to A^i(X)$ such that for all $W \subseteq Y$ a smooth subvariety with $i = \operatorname{codim}_Y W = \operatorname{codim}_X(f^{-1}(W))$, we have $f^*([W]) = [f^{-1}(W)]$. This extends to a ring homomorphism $f^* : A^*(Y) \to A^*(X)$ called the *pullback* of f.

REMARK A.3. The degree of the field extension in the definition of f_* is the degree of the covering of $f(V)$ by V. In particular, if $i : X \to Y$ is a closed immersion, then $i_*([X]) = [X]$.

PROPOSITION A.4 (Projection Formula, [16, Theorem 1.23(b)]). *If X and Y are smooth projective varieties, $f : X \to Y$ is a morphism, and $[C] \in A_i(X)$ and $[D] \in A^j(Y)$ are cycle classes, then*
$$[D] f_*([C]) = f_*(f^*([D])[C]) \in A_{i-j}(Y).$$

DEFINITION A.5 ([16, Theorem 5.3]). Let X be a smooth projective variety of dimension n and let $\mathscr{B}$ be an almost generated vector bundle over X, see Definiton 5.2. There exist unique classes $c_i(\mathscr{B}) \in A^i(X)$ for $i \in [n]$ called the *Chern classes* of $\mathscr{B}$, depending only on the isomorphism class of $\mathscr{B}$, satisfying the following axioms:

(i) If r is the rank of $\mathscr{B}$, then $c_i(\mathscr{B}) = 0$ for all $i > r$.

(ii) If $\sigma_0, ..., \sigma_{r-i} \in \Gamma(\mathscr{B})$ are global sections and their degeneracy locus $Z(\sigma_0, ..., \sigma_{r-i}) \subseteq X$ has codimension i in X, then $c_i(\mathscr{B}) = [Z(\sigma_0, ..., \sigma_{r-i})]$.

(iii) The *Chern polynomial* of $\mathscr{B}$ is $C(t, \mathscr{B}) = 1 + \sum_{i=1}^r c_i(\mathscr{B}) t^i$. If $0 \to \mathscr{B} \to \mathscr{B}' \to \mathscr{B}'' \to 0$ is an exact sequence of vector bundles over X, then
$$C(t, \mathscr{B}') = C(t, \mathscr{B}) C(t, \mathscr{B}'').$$

(iv) If Y is a smooth projective variety and $f : Y \to X$ a morphism, $c_i(f^*\mathscr{B}) = f^*(c_i(\mathscr{B}))$.

If $r = n$, then $c_n(\mathscr{B}) \in A^n(X)$ so $c_n(\mathscr{B}) = \nu(\mathscr{B})[p]$ for some integer $\nu(\mathscr{B})$ called the *top Chern number* of $\mathscr{B}$, where $[p] \in A^n(X)$ is the class of a point $p \in X$.

REMARK A.6. [16, Theorem 5.3] gives a definition of Chern classes for any vector bundle $\mathscr{B}$ over X, not just those that are almost generated. However, when $\mathscr{B}$ is almost generated, then part (a) of that result is redundant since in this case it is already covered by part (b), due to [16, Lemma 5.2(b)]. We replace [16, Theorem 5.3(a)] with our Definition A.5(a), which is also a redundant axiom but helps clarify the properties of Chern classes.

DEFINITION A.7 ([21, Remark 3.2.3, Example 3.2.3]). The *Chern roots* of $\mathscr{B}$ are the formal variables $\xi_i(\mathscr{B})$ in the formal factorization of the Chern polynomial:

$$C(t, \mathscr{B}) = \prod_{i=1}^{r} (1 + \xi_i(\mathscr{B})t).$$

The *Chern character* of $\mathscr{B}$ is $\mathrm{ch}(\mathscr{B}) = \sum_{i=1}^{r} \exp(\xi_j(\mathscr{B}))$, where $\exp(\alpha) = \sum_{k=0}^{\infty} \frac{1}{k!}\alpha^k$ is a formal sum in the formal variable α.

From Definitions A.5 and A.7, one can obtain the following properties.

PROPOSITION A.8 ([21, Remark 3.2.3, Example 3.2.3]). *Let X be a smooth projective variety and $\mathscr{B}$ and $\mathscr{B}'$ be vector bundles over X.*

- *(a) If $\mathscr{B}$ is the trivial bundle, then $C(t, \mathscr{B}) = 1$.*
- *(b) The Chern polynomial of $\mathscr{B}$ and its dual are related by $C(t, \mathscr{B}^\vee) = C(-t, \mathscr{B})$.*
- *(c) The Chern character satisfies $\mathrm{ch}(\mathscr{B} \otimes \mathscr{B}') = \mathrm{ch}(\mathscr{B})\,\mathrm{ch}(\mathscr{B}')$.*

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