Effective Whitney Stratification of Real Algebraic Varieties

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ABSTRACT. We describe an algorithm to compute Whitney stratifications of real algebraic varieties. The basic idea is to first stratify the complexified version of the given real variety using conormal techniques, and then to show that the resulting stratifications admit a description using only real polynomials. This method also extends to stratification problems involving certain basic semialgebraic sets as well as certain algebraic maps. One of the map stratification algorithms described here yields a new method for solving the real root classification problem.

1. Introduction

A pair (M, N) of smooth submanifolds of $\mathbb{R}^n$ satisfies Whitney's **Condition (B)** if the following property holds at every point $q \in N$. Given any pair of sequences $\{p_k\} \subset M$ and $\{q_k\} \subset N$ with $\lim p_k = q = \lim q_k$, if the limiting tangent space and the limiting secant line

$$T := \lim_{k \to \infty} T_{p_k} M \quad \text{and} \quad \ell := \lim_{k \to \infty} [p_k, q_k]$$

both exist, then $\ell \subset T$. A **Whitney stratification** of a subset $X \subset \mathbb{R}^n$ is any locally-finite decomposition of $X = \coprod_\alpha M_\alpha$ into smooth, connected nonempty manifolds $M_\alpha \subset X$ called *strata*, so that every pair (M_α, M_β) satisfies Condition (B). The main contribution of this note is a practical algorithm for constructing Whitney stratifications of real algebraic varieties. The existence of such stratifications dates back to the work of Whitney [**23**] — every real algebraic variety X admits a Whitney stratification $\coprod_\alpha M_\alpha$ such that for each dimension $i \geq 0$, the union $X_i \subset \mathbb{R}^n$ of all strata of dimension $\leq i$ is a subvariety of X [**18**].

From real to complex and back. In prior work, we used conormal spaces and primary decomposition to algorithmically stratify complex algebraic varieties [**14**]. In the introductory remarks to that paper, we highlighted the lack of Gröbner basis techniques over $\mathbb{R}$ as a primary obstacle to performing similar stratifications for real algebraic varieties and semialgebraic sets. We overcome this obstacle in Section 2 of this paper by constructing, for any real variety $X \subset \mathbb{R}^n$, the corresponding complex variety $X(\mathbb{C}) \subset \mathbb{C}^n$ — this is precisely the vanishing locus of the defining polynomials of X, treated as an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. The key insight is that the subvarieties arising from a stratification of $X(\mathbb{C})$ produced by the methods of [**14**] are also generated by real polynomials; and the real varieties defined by those polynomials constitute a valid Whitney stratification of X.

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Stratifying real algebraic maps. A stratification of a real algebraic map $f : X \to Y$ is a pair of Whitney stratifications of X and Y so that f sends each stratum $M \subset X$ smoothly and submersively to a single stratum $N \subset Y$. Whenever f is proper (i.e., if X is compact) then the restriction of f to $f^{-1}(N)$ forms a locally trivial fiber bundle over N, which in particular implies that the stratified homeomorphism type of the fiber $f^{-1}(y)$ is independent of the choice of $y \in N$. Our second contribution, carried out in Section 3, is to describe an algorithm for stratifying any given f. As with real varieties, the key step is to first consider a complexified version $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ of the morphism, and to then employ the methods of [**14**].

Stratifying full semialgebraic sets. The ability to algorithmically stratify real varieties also allows us to produce Whitney stratifications of certain basic semialgebraic sets. In Section 4, we consider *full* semialgebraic sets B of the form $X \cap C$, where X is a real algebraic variety and $C \subset \mathbb{R}^n$ is a region carved out by polynomial inequalities of the form $f_i(x) \geq 0$ whose interior is an open n-dimensional submanifold of $\mathbb{R}^n$. Let Y_C be the real hypersurface defined by the vanishing of $\prod_i f_i$. Our third contribution here is to describe a mechanism for inducing a Whitney stratification of B from Whitney stratifications of the real varieties X and $X \cap Y_C$. We note that full semialgebraic sets arise rather frequently in applications, so we expect their stratifications to be useful across a broad spectrum of practical problems.

Dominant maps and real root classification. The real root classification problem seeks to describe how the number of real roots of a parametric polynomial system varies as a function of the parameters. This problem appears in a variety of applied contexts, including chemical reaction networks [**7, 9**], medical imaging [**2**], computer vision [**8**], kinematics and robotics [**5, 11**], ordinary differential equations [**4**], and quadrature domains [**1**]. In Section 5 we describe an algorithm for stratifying dominant polynomial maps $f : X \to Y$ between real varieties of the same dimension; we are able to partially recreate the stratified fiber bundle property of proper maps by carefully analysing and decomposing the locus of points at which f fails to be proper. In Section 6, we show how dominant map stratifications help solve the real root classification problem by decomposing the parameter space into certain strata over which the number of roots is locally constant.

2. Stratifying Real Algebraic Varieties

Let $\mathbb{R}[x_1, \ldots, x_n]$ be the ring of real polynomials in n indeterminates, and fix a radical ideal I of this ring. By definition, the vanishing locus $X := \mathbf{V}(I)$ constitutes a real algebraic subvariety of $\mathbb{R}^n$. Since $\mathbb{R}[x_1, \ldots, x_n]$ is a subring of $\mathbb{C}[x_1, \ldots, x_n]$, the ideal I similarly defines a complex algebraic subvariety $\mathbf{V}_{\mathbb{C}}(I)$ of $\mathbb{C}^n$, which we will denote by $X(\mathbb{C})$. Let X_{reg} denote the manifold of smooth points in X. In this section $\dim_{\mathbb{R}}(X)$ will denote the dimension of manifold X_{reg} and $\dim_{\mathbb{C}}(X(\mathbb{C}))$ will denote the dimension of the manifold $(X(\mathbb{C}))_{\text{reg}}$. Our immediate goal here is to show that certain Whitney stratifications of $X(\mathbb{C})$ induce Whitney stratifications of X. We let $\iota : \mathbb{R}^n \hookrightarrow \mathbb{C}^n$ be the embedding of real points in complex Euclidean space.

LEMMA 2.1. *Let $X \subset \mathbb{R}^n$ be a real algebraic variety.*

(1) *The embedding* ι *identifies* X *with the real points of* $X(\mathbb{C})$.
(2) *Assume that* $\dim_{\mathbb{R}} X$ *equals* $\dim_{\mathbb{C}} X(\mathbb{C})$. *If* $\iota(p)$ *is a smooth point of* $X(\mathbb{C})$ *for some* $p \in X$, *then* p *is a smooth point of* X.

PROOF. The first assertion is a tautology. Turning to the second assertion, set $d := \dim_{\mathbb{C}} X(\mathbb{C}) = \dim_{\mathbb{R}} X$. Since the roots of real polynomials occur in complex conjugate pairs, the variety $X(\mathbb{C})$ is invariant under complex conjugation and $\iota(X)$ equals the fixed point set of this conjugation. Noting that $\iota(p)$ is a smooth point by assumption, the tangent space $T_{\iota(p)}X(\mathbb{C})$ exists and has complex dimension d; this tangent space also inherits invariance under complex conjugation. Thus, $T_{\iota(p)}X(\mathbb{C})$ is the complexification of a real d-dimensional vector space V whose elements consist of all real tangent vectors at $\iota(p)$; this V is evidently isomorphic to T_pX, as desired. □

Every finite descending chain $I_\bullet$ of radical R-ideals

$$I_0 \rhd I_1 \rhd \cdots \rhd I_m = I$$

produces an ascending flag $X_\bullet := \mathbf{V}(I_\bullet)$ of subvarieties of X:

$$X_0 \subset X_1 \subset \cdots \subset X_m = X.$$

The next result shows that successive differences of $X_\bullet$ inherit a smooth manifold structure from the successive differences of $X_\bullet(\mathbb{C})$.

PROPOSITION 2.2. *Let* $W \subset Z$ *be a pair of real algebraic varieties in* $\mathbb{R}^n$. *If the difference* $Z(\mathbb{C}) - W(\mathbb{C})$ *is either empty or a smooth i-dimensional complex manifold, then* $M := (Z - W)$ *is either empty or a smooth i-dimensional real manifold.*

PROOF. There are two cases to consider — either the image $\iota(Z)$ lies entirely within the singular locus $Z(\mathbb{C})_{\text{sing}}$, or there exists some $p \in M$ with $\iota(p) \in Z(\mathbb{C})_{\text{reg}}$. In the first case, since $Z(\mathbb{C}) - W(\mathbb{C})$ is smooth, we know that $Z(\mathbb{C})_{\text{sing}}$ lies entirely within $W(\mathbb{C})$ and hence that $\iota(Z) \subset \iota(W)$; but since we have assumed $W \subset Z$, we must have $W = Z$, whence M is empty. On the other hand, let p be a point in $(Z - W)$ for which $\iota(p)$ is a smooth point of $Z(\mathbb{C})$. We may safely assume that the generating ideal of Z is prime in $\mathbb{C}[x_1, \ldots, x_n]$ by passing to the irreducible component which contains $\iota(p)$. It now follows from [**19**, Theorem 12.6.1] or [**12**, Theorem 2.3] that $(Z - W)$ has dimension i. Finally, Lemma 2.1 ensures that $(Z - W)$ is a smooth real i-manifold. □

It follows from the above result that if $X_\bullet(\mathbb{C})$ is a Whitney stratification of $X(\mathbb{C})$, then the successive differences of $X_\bullet$ are either empty or smooth manifolds of the expected dimension. We show below these successive differences also satisfy Condition (B).

THEOREM 2.3. *Let* $I_\bullet$ *be a descending chain of radical ideals in* $\mathbb{R}[x_1, \ldots, x_n]$. *If the flag* $X_\bullet(\mathbb{C}) := \mathbf{V}_{\mathbb{C}}(I_\bullet)$ *constitutes a Whitney stratification of* $X(\mathbb{C})$, *then the corresponding flag* $X_\bullet := \mathbf{V}(I_\bullet)$ *yields a Whitney stratification of* X.

PROOF. Consider a non-empty connected component $M \subset S_i$, and let $V \subset X_i$ be the irreducible component which contains M. Similarly, let $W \subset X_i(\mathbb{C})$ be the irreducible component which contains $\iota(M)$ and define $M_{\mathbb{C}} := W - X_{i-1}(\mathbb{C})$. We note that $\iota(M)$ forms an open subset of $M_{\mathbb{C}}$, which must in turn by an i-stratum of $X(\mathbb{C})$. Similarly, consider a nonempty connected $N \subset S_j$ with $i > j$ and analogously define $N_{\mathbb{C}} \subset X_j(\mathbb{C})$. We will show that the pair (M, N) satisfies Condition (B). To this end, consider a point $q \in N$ along with sequences $\{p_k\} \subset M$ and $\{q_k\} \subset N$ which converge to q. Letting ℓ_k denote the secant line $[p_k, q_k]$ and T_k the tangent plane $T_{p_k}M$, we assume further that the limits $\ell = \lim \ell_k$ and $T = \lim T_k$ both exist.

Let $\ell_k(\mathbb{C})$ be the secant line $[\iota(p_k), \iota(q_k)]$ in $\mathbb{C}^n$ and $T_k(\mathbb{C})$ the tangent space $T_{\iota(p_k)}M_{\mathbb{C}}$. Since $\iota(p_k)$ and $\iota(q_k)$ are real points for all k, the the linear equations defining both the secant lines $\ell_k(\mathbb{C})$ and the tangent space $T_k(\mathbb{C})$ as varieties are exactly the same as those defining ℓ_k and T_k, respectively. Thus, the limits $\ell(\mathbb{C})$ and $T(\mathbb{C})$ both exist because the corresponding real limits exist – one may view these as limits of real sequences inside a complex Grassmannian – and they are defined by the same algebraic equations as their counterparts ℓ and T. By definition of secant lines, the image $\iota(w)$ of any $w \in \ell$ is a real point of $\ell(\mathbb{C})$. Since the pair $(M_{\mathbb{C}}, N_{\mathbb{C}})$ satisfies Condition (B) by assumption, we know that $\ell(\mathbb{C}) \subset T(\mathbb{C})$, whence $\iota(w)$ must be a real point of $T(\mathbb{C})$. Since $\iota(M)$ is an open subset of $M_{\mathbb{C}}$, we have $T_{\iota(p_k)}M_{\mathbb{C}} = T_{\iota(p_k)}\iota(M)$ for all k; and by the proof of Lemma 2.1, the real points of $T_{\iota(p_k)}\iota(M)$ are identified with $T_{q_k}N$. Thus, $\iota(T)$ contains all the real points of $T(\mathbb{C})$, including $\iota(w)$. Since ι is injective, we have $w \in T$ as desired. ☐

COROLLARY 2.4. *Let X be an algebraic variety in $\mathbb{R}^n$. The* **WhitStrat** *algorithm of* [**14**]*, when applied to $X(\mathbb{C})$, produces a Whitney stratification of X.*

PROOF. The **WhitStrat** algorithm performs three types of operations: ideal addition, Gröbner basis computation, and primary decomposition. Each of these operations leaves the coefficient field of all intermediate polynomials unchanged. ☐

3. Stratifying Real Algebraic Morphisms

Maps between Whitney stratified spaces are typically required to satisfy additional criteria beyond smoothly sending strata to strata — see [**3**, Def 3.5.1] or [**10**, Part I, Ch 1.7] for instance.

DEFINITION 3.1. Let $\mathscr{X}_\bullet$ and $\mathscr{Y}_\bullet$ be Whitney stratifications of topological spaces $\mathscr{X}$ and $\mathscr{Y}$. A continuous function $\phi : \mathscr{X} \to \mathscr{Y}$ is **stratified** with respect to $\mathscr{X}_\bullet$ and $\mathscr{Y}_\bullet$ if for each stratum $M \subset \mathscr{X}$ there exists a a stratum $N \subset \mathscr{Y}$ satisfying two requirements:

(1) the image $\phi(M)$ is wholly contained in N; and moreover,
(2) the restricted map $\phi|_M : M \to N$ is a smooth submersion.[1]

The pair $(\mathscr{X}_\bullet, \mathscr{Y}_\bullet)$ is called a stratification of ϕ.

REMARK 3.2. The second requirement of Definition 3.1 ensures the following crucial property via Thom's first isotopy lemma [**20**, Prop 11.1]. If ϕ is a *proper map* – namely, if the inverse image of every compact subset of Y is compact in X – then for every stratum $N \subset Y$, the restriction of ϕ forms a locally trivial fiber bundle from $\phi^{-1}(N)$ to N. In general, the fibers are not guaranteed to be smooth.

[1]Explicitly, its derivative $d(\phi|_S)_x : T_xM \to T_{\phi(x)}N$ is surjective at each point x in M.

Consider algebraic varieties $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and let $f : X \to Y$ be an algebraic morphism — concretely, this amounts to an m-tuple of real polynomials

$$\Big(f_1(x_1, \ldots, x_n),\ f_2(x_1, \ldots, x_n),\ \ldots,\ f_m(x_1, \ldots, x_n)\Big)$$

whose evaluation at a point of X yields a point of Y. Since each f_i is automatically a complex polynomial, there is an evident morphism $f_\mathbb{C} : X(\mathbb{C}) \to Y(\mathbb{C})$ of complex algebraic varieties. Let $I_\bullet$ and $J_\bullet$ be descending chains of radical ideals in $\mathbb{R}[x_1, \ldots, x_m]$ and $\mathbb{R}[y_1, \ldots, y_m]$ respectively so that $X_\bullet(\mathbb{C}) := \mathbf{V}_\mathbb{C}(I_\bullet)$ and $Y_\bullet(\mathbb{C}) := \mathbf{V}_\mathbb{C}(J_\bullet)$ constitute Whitney stratifications of $X(\mathbb{C})$ and $Y(\mathbb{C})$ respectively. It follows from Theorem 2.3 that $X_\bullet := \mathbf{V}(I_\bullet)$ is a Whitney stratification of X while $Y_\bullet := \mathbf{V}(J_\bullet)$ is a Whitney stratification of Y.

THEOREM 3.3. *If $f_\mathbb{C}$ is stratified with respect to $X_\bullet(\mathbb{C})$ and $Y_\bullet(\mathbb{C})$, then f is stratified with respect to $X_\bullet$ and $Y_\bullet$.*

PROOF. Let $M \subset X$ be a nonempty connected component of the i-stratum $X_i - X_{i-1}$, and let $M_\mathbb{C}$ be the i-stratum of $X_\bullet(\mathbb{C})$ which contains $\iota(M)$. By definition, the image $f_\mathbb{C}(M_\mathbb{C})$ contains $f(M)$ in its locus of real points. Since $f_\mathbb{C}$ is stratified with respect to $X_\bullet(\mathbb{C})$ and $Y_\bullet(\mathbb{C})$, the first requirement of Definition 3.1 guarantees the existence of a single stratum $N_\mathbb{C} \subset Y$ which contains $f_\mathbb{C}(M_\mathbb{C})$. Thus, $f(M)$ lies in the locus of real points of $N_\mathbb{C}$. Letting N denote the stratum of $Y_\bullet$ corresponding to $N_\mathbb{C}$, we know that the real locus of $N_\mathbb{C}$ equals N, whence we obtain $f(M) \subset N$ and it remains to show that the restriction of f to M yields a submersion. Let x be any point of M, and note that $\iota(x)$ lies in $M_\mathbb{C}$. Since $f_\mathbb{C}|_{M_\mathbb{C}}$ is a submersion, its derivative at $\iota(x)$ is a surjective linear map from the tangent space to $M_\mathbb{C}$ at $\iota(x)$ to the tangent space to $N_\mathbb{C}$ at $f_\mathbb{C} \circ \iota(x)$. But by construction, $f_\mathbb{C} \circ \iota$ equals f. Thus, we have

$$\operatorname{rank}_\mathbb{C} \left(d f_\mathbb{C}|_{M_\mathbb{C}}(\iota(x))\right) = \dim_\mathbb{C} N_\mathbb{C}.$$

To conclude the argument, we note that the derivative arising on the left side of the above equality may be represented by the Jacobian matrix of f at x, and the rank of this matrix is preserved under field extension to $\mathbb{C}$. On the other hand, by Proposition 2.2 we know that the complex dimension of $N_\mathbb{C}$ equals the real dimension of N. Thus, our equality simplifies to

$$\operatorname{rank}_\mathbb{R} \left(d f|_M(x)\right) = \dim_\mathbb{R} N,$$

as desired. □

Assume that a morphism $f : X \to Y$ has been stratified as described in Theorem 3.3. It is readily checked that the image $f(\overline{M})$ of the closure of a stratum $M \subset X$ is not an algebraic subvariety of Y in general — the best that one can expect is that $f(\overline{M})$ will be semialgebraic. It is important to note, in the context of the above theorem, that we do not obtain semialgebraic descriptions of such images.

4. Stratifying Full Semialgebraic Sets

A (basic, closed) *semialgebraic set* is any subset $B \subset \mathbb{R}^n$ which can be expressed as an intersection of the form $B := X \cap C$ where X is a real algebraic subvariety of $\mathbb{R}^n$, while the set C, called an *inequality locus*, is given as follows:

$$C := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } 0 \leq i \leq k\}. \tag{1}$$

Here the g_i's are a finite collection of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. By convention, when the number of inequalities k equals zero, we have $C = \mathbb{R}^n$. Thus, every algebraic variety is automatically a semialgebraic set in the above sense. The sets X and C are not uniquely determined for a given B in general — we may, for instance, safely remove any polynomial generator $f : \mathbb{R}^n \to \mathbb{R}$ from the defining ideal of X while adding $f \geq 0$ and $-f \geq 0$ to the inequality locus. It is therefore customary to omit X entirely and simply define B as the set of points which satisfy a collection of polynomial inequalities. We find it convenient to write $B = X \cap C$ here because this allows us to highlight a relevant sub-class of semialgebraic sets.

DEFINITION 4.1. A semialgebraic set $B \subset \mathbb{R}^n$ is called a **full** if it admits an inequality locus C of the form (1), with the additional requirement that its subset

$$C^\circ := \{x \in \mathbb{R}^n \mid g_i(x) > 0 \text{ for } 0 \leq i \leq k\}$$

is an n-dimensional smooth manifold whose closure equals C.

Given an inequality locus C of a full semialgebraic set, we call C° its interior and define its boundary as the difference

$$\partial C := C - C^\circ.$$

This boundary is a semialgebraic subset of the real algebraic variety $Y_C := \mathbf{V}(\prod_1^k g_i)$. We adopt the usual convention that the product over the empty set equals 1, which forces $\partial C = Y_C = \varnothing$ when $k = 0$. We recall that a Whitney stratification $X_\bullet$ of X is **subordinate** to a flag $F_0 \subset F_1 \subset \cdots \subset F_k = X$ if for each $X_\bullet$-stratum $S \subset X$ there exists some j satisfying $S \subset (F_j - F_{j-1})$, see [14, Definition 5.1] for additional details. Our next result establishes that every full semialgebraic set $B = X \cap C$ inherits a Whitney stratification from Whitney stratifications of the real algebraic varieties X and $X \cap Y_C$.

THEOREM 4.2. *Let $B = X \cap C$ be a full semialgebraic set, and let $X_\bullet$ be a Whitney stratification of X. If $Y_\bullet$ is a Whitney stratification of $X \cap Y_C$ which is subordinate to the flag $X_\bullet \cap Y_C$, then setting*

$$B_i := (X_i \cup Y_i) \cap C$$

produces a Whitney stratification of B.

PROOF. Since B is full, we know that the interior C° of its inequality locus is a smooth open n-dimensional submanifold of $\mathbb{R}^n$. Therefore, the intersections $X_i \cap C^\circ$ form a Whitney stratification $X'_\bullet$ of $X \cap C^\circ$. Let $Y'_\bullet$ be the subset of $Y_\bullet$-strata which intersect ∂C. Since C is the disjoint union of C° and ∂C, it follows that the union of $X'_\bullet$-strata and $Y'_\bullet$-strata partitions B. It remains to check that Condition (B) holds for those strata pairs (M, N) of this union for which N intersects the closure of M. There are now three cases to consider, of which the two easy ones are handled as follows:

(1) if both M and N are strata of $X'_\bullet$, then Condition (B) holds because both are full-dimensional open subsets of $X_\bullet$-strata by construction, and $X_\bullet$ is assumed to be a Whitney stratification.

(2) if M is a $Y'_\bullet$-stratum, then the fact that N intersects the closure of M forces N to be contained in $X \cap \partial C$, since both X and ∂C are closed subsets of $\mathbb{R}^n$. Thus, N must also be a $Y'_\bullet$-stratum in which case Condition (B) holds because $Y'_\bullet$ is Whitney.

Turning now to the third case, assume that M is an $X'_\bullet$-stratum and N is a $Y'_\bullet$-stratum. By construction, M must be (a connected component of) the intersection $M_* \cap C^\circ$ for some $X_\bullet$-stratum M_*. Since C° is n-dimensional, the tangent spaces T_xM and T_xM_* coincide for every x in M. Fix a point $p \in N$, and let N_* be the unique $X_\bullet$-stratum containing p in its interior. Since $Y_\bullet$ is chosen subordinate to $X_\bullet \cap Y_C$, the $Y_\bullet$-strata are refinements of $X_\bullet$-strata, so N must be obtained by removing some (possibly empty) set from $N^* \cap \partial C$. It follows that N is a subset of N^* in a small ball around p. Finally, (M, N) must satisfy Condition (B) at p because (M_*, N_*) satisfy Condition (B) at p. □

The stratifications obtained in Theorem 4.2 provide a complete description of the flag B_i. Using the techniques of [**12**, **17**, **21**], one can perform various fundamental algorithmic tasks involving such strata. These include testing whether the i-stratum $B_i - B_{i-1}$ is empty for each i, and sampling points from the non-empty strata.

5. Stratifying Dominant Maps between Equidimensional Varieties

Let $X \subset \mathbb{K}^n$ and $Y \subset \mathbb{K}^m$ be algebraic varieties defined by ideals I_X and I_Y over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathbb{K}[X]$ denote the coordinate ring $\mathbb{K}[x_1, \ldots, x_n]/I_X$ and similarly for Y; any morphism of varieties $f : X \to Y$ canonically induces a contravariant ring homomorphism $f^* : \mathbb{K}[Y] \to \mathbb{K}[X]$.

DEFINITION 5.1. Let $f : X \to Y$ be a morphism of algebraic varieties over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(1) we say that f is **dominant** if f^* is a monomorphism (or equivalently, if the image $f(X)$ is dense in Y.
(2) we say that f is **finite** if it is dominant, and moreover, if f^* gives $\mathbb{K}[X]$ the structure of an integral extension of $\mathbb{K}[Y]$.

It is a classical result [**13**, Exercise II.4.1] that if $f : X \to Y$ is finite in the above sense, then it is also a proper map for any field (see Remark 3.2 for the significance of this result in our context). Over $\mathbb{K} = \mathbb{C}$, a dominant morphism is finite if and only if it is proper .

Throughout this section, $f : X \to Y$ will denote a morphism between real algebraic varieties of the same dimension d; we will further require that the map $f_\mathbb{C} : X(\mathbb{C}) \to Y(\mathbb{C})$ is dominant and that $\dim_\mathbb{C} X(\mathbb{C}) = \dim_\mathbb{C} Y(\mathbb{C}) = d$.

DEFINITION 5.2. The **Jelonek set** of f is the subset $\mathbf{Jel}(f) \subset Y(\mathbb{C})$ consisting of all points y for which there exists a sequence $\{x_n\} \subset X(\mathbb{C})$ satisfying both

$$\lim_{n\to\infty} |x_n| = \infty \quad \text{and} \quad \lim_{n\to\infty} f_\mathbb{C}(x_n) = y.$$

It is shown in [**15**] that $\mathbf{Jel}(f)$ is either empty or an algebraic hypersurface of $Y(\mathbb{C})$; a Gröbner basis algorithm for computing the Jelonek set is given in [**22**]. It follows from this algorithm that if the Jelonek set is non-empty, then it is defined by a polynomial with real coefficients.

Finally, it is shown in [15] that $\mathbf{Jel}(f)$ is precisely the locus of points at which f fails to be finite. Thus, if we define

$$V(\mathbb{C}) := \mathbf{Jel}(f) \quad \text{and} \quad W(\mathbb{C}) := \overline{f^{-1}(V)},$$

then the restriction of f forms a proper map $(X(\mathbb{C}) - W(\mathbb{C})) \to (Y(\mathbb{C}) - V(\mathbb{C}))$ — see [16, Proposition 6.1] for details. Note that the polynomials defining $V(\mathbb{C})$ and $W(\mathbb{C})$ are real; take V to be the real zero set of the polynomials defining $V(\mathbb{C})$, and similarly let W be the real zero set of the polynomials defining $W(\mathbb{C})$. It follows immediately that the restriction of f to the difference $(X - W)$ constitutes a proper map to the difference $(Y - V)$.

DEFINITION 5.3. The **Jelonek flag** of $f : X \to Y$ is a pair $(W_\bullet, V_\bullet)$ of flags, both of length $d = \dim X = \dim Y$:

$$\varnothing = W_{-1} \subset W_0 \subset \cdots \subset W_d = X$$
$$\varnothing = V_{-1} \subset V_0 \subset \cdots \subset V_d = Y,$$

defined via reverse-induction on $i \in \{d-1, d-2, \ldots, 1, 0\}$ as follows. Starting with V_{d-1} as the real points of $\mathbf{Jel}(f)$, we

(1) let W_i be $\overline{f^{-1}(V_i)}$, and
(2) let V_{i-1} be the real points in $\mathbf{Jel}(f|_{W_i} : W_i \to V_i)$.

By construction of the Jelonek flag, at each i we have the following alternative:

(1) **if** $V_i(\mathbb{C})$ **is nonempty**, then $\dim V_i(\mathbb{C}) = W_i(\mathbb{C}) = i$ and the restriction of f forms a proper map $(V_i - V_{i-1}) \to (W_i - W_{i-1})$; otherwise,
(2) **if** $V_i(\mathbb{C})$ **is empty**, then $f|_{W_i} : W_i \to V_i$ is not dominant; on the other hand, $f|_{W_{i-1}}$ is dominant, but with $\dim W_{i-1}(\mathbb{C}) > \dim V_{i-1}(\mathbb{C})$.

THEOREM 5.4. *Let $(W_\bullet, V_\bullet)$ be the Jelonek flag of $f : X \to Y$. Assume that $X_\bullet$ is a Whitney stratification of X subordinate $W_\bullet$ and that $Y_\bullet$ is a Whitney stratification of Y subordinate to $V_\bullet$. Whenever* $\dim V_i = \dim W_i$ *holds, we have that for every $Y_\bullet$-stratum $R \subset (V_i - V_{i-1})$, the map $f|_{f^{-1}(R)} : f^{-1}(R) \to R$ is a locally trivial fiber bundle.*

PROOF. Since the stratification $Y_\bullet$ of Y is subordinate to $V_\bullet$ then for all strata R we have that $R \subset V_i - V_{i-1}$ for some i, and also since $X_\bullet$ is subordinate to $W_\bullet$ we also have $f^{-1}(R) \subset W_i - W_{i-1}$. If $\dim(V_i) = \dim(W_i)$ holds, then the map $f : (W_i - W_{i-1}) \to (V_i - V_{i-1})$ is proper. An appeal to the semialgebraic version of Thom's first isotopy lemma [6, Theorem 1] achieves the desired result. □

6. Real Root Classification

In this section we explore how the methods developed in the previous Section can be used to study the *real root classification problem*. To this end, fix integers $m, n \geq 1$ and define the subset

$$P \subset (\mathbb{R}[c_1, \ldots, c_m])\,[x_1, \ldots, x_n].$$

consisting of all polynomials which have the form

$$f(x,c) = \sum_{j=1}^{k} c_j \cdot x_1^{a_{1,j}} \cdots x_n^{a_{n,j}} + g(x),$$

where $\{a_{i,j}\}$ is some $k \times n$ matrix of non-negative integers and $g(x)$ is a polynomial in $\mathbb{R}[x_1, \dots, x_n]$.

DEFINITION 6.1. Consider polynomials $f_1(x,c), \dots, f_n(x,c)$ in P, and suppose that the system

$$\{f_i(x,c) = 0 \mid 1 \leq i \leq n\} \tag{2}$$

has finitely many complex solutions in $\mathbb{C}^n$, for a generically chosen parameter $c \in \mathbb{R}^m$. The **real root classification problem** seeks a decomposition $\mathbb{R}^m = \coprod_j M_j$ so that either the number of real solutions to $\{f_i(x,c^*) = 0\}$ is locally constant across $c^* \in M_j$, or the system has infinitely many complex solutions.

Henceforth, we treat the polynomials $f_1, \dots, f_n$ in P as polynomials in $\mathbb{R}[x,c]$ and consider the variety $X = \mathbf{V}(f_1, \dots, f_n)$ in $\mathbb{R}^n \times \mathbb{R}^m$. Note that $\dim(X) = m$ by the assumption that for a fixed generic parameter value the system (2) has finitely many complex solutions. Hence, if we take $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ to be the coordinate projection onto the last m coordinates, then the resulting map $\pi : X \to \mathbb{R}^m$ is dominant and $\dim(X) = m$. Thus, π is a dominant map between varities of the same dimension. Note that for a point $q \in Y$ the fiber $\pi^{-1}(q)$ consists of a set of points of the form (x,q) in $\mathbb{R}^n \times \mathbb{R}^m$ where the x values are exactly the solutions in $\mathbb{R}^n$ to the system (2) for the choice of parameters $c = q$; also note that for generic q the set $\pi^{-1}(q)$ is finite.

THEOREM 6.2. *Set $Y = \mathbb{R}^m$ and consider the dominant projection map $\pi : X \to Y$ defined above. Let $(W_\bullet, V_\bullet)$ be the Jelonek flag of π. If $(X_\bullet, Y_\bullet)$ is a stratification of π in the sense of Definition 3.1, with $X_\bullet$ subordinate to $W_\bullet$ and $Y_\bullet$ subordinate to $V_\bullet$, then for any $Y_\bullet$-stratum N, we have that either the number of real points in $\pi^{-1}(q)$ is fixed and independent of q or $\pi^{-1}(q)$ has infinitely many complex points.*

PROOF. Since $Y_\bullet$ is subordinate to $V_\bullet$ we have that $N \subset (V_i - V_{i-1})$ for some i. There are two cases to consider:

(1) if $\pi : W_i \to V_i$ is dominant with $\dim(W_i) > \dim(V_i)$, then we have $\dim(f^{-1}(Z)) > \dim(Z)$ for any subvariety $Z \subset V_i$. Hence, for any $q \in N$ the fiber $f^{-1}(q)$ consists of infinitely many complex points.
(2) if $\dim(V_i) = \dim(W_i)$, then the conclusion follows immediately from Theorem 5.4 since whenever the fibers are zero dimensional and the number of real points is exactly the number of connected components.

Thus, any Jelonek-subordinate stratification of $\pi : X \to Y$ directly solves the real root classification problem □

Examples. We conclude with two simple examples of real root classification arising from dominant map stratification.

EXAMPLE 6.3. Consider the quadratic equation:

$$ax^2 + bx + c = 0 \tag{3}$$

where we think of a, b, c as real parameters. We wish to classify its real solutions, to do this we consider the variety $X = \mathbf{V}(ax^2 + bx + c)$ in $\mathbb{R}^4$ and the projection map $\pi : \mathbb{R}^4 \to \mathbb{R}^3$

onto $Y = \mathbb{R}^3$ specified by $(x, a, b, c) \mapsto (a, b, c)$. A stratification of π as in Theorem 6.2 is given by:

$$X_\bullet = \mathbf{V}(ax^2 + bx + c) \supset \left(\mathbf{V}(b^2 - 4ac,\, ax^2 + bx + c) \cup \mathbf{V}(a,\, bx + c)\right) \supset \mathbf{V}(a, b, c)$$

and

$$Y_\bullet = \mathbb{R}^3 \supset \left(\mathbf{V}(b^2 - 4ac) \cup \mathbf{V}(a)\right) \supset \mathbf{V}(a, b) \supset \mathbf{V}(a, b, c).$$

The number of real solutions of $ax^2 + bx + c = 0$ is locally constant on every stratum of Y.

First consider the strata of dimension 3 arising from $M_3 = \mathbb{R}^3 - \left(\mathbf{V}(b^2 - 4ac) \cup \mathbf{V}(a)\right)$; M_3 has 4 connected components, one representative point of each of these is: $(-1, 1, 1) \in S_1$, $(-1, 1, -1) \in S_2$, $(1, 1, 1) \in S_3$, $(1, 1, -1) \in S_4$. The boundary of the 4 connected components of M_3 is illustrated below in Figure 1. Using the representative points we see that (3) has two real solutions for all coefficients $(a, b, c) \in S_1$ and $(a, b, c) \in S_4$, and no real solutions for $(a, b, c) \in S_2$ and for $(a, b, c) \in S_3$. Note that our algorithm does not produce the semi-algebraic description of the sets S_i, e.g. we do not compute that $S_1 = \{(a, b, c) \in \mathbb{R}^3 \mid b^2 - 4ac > 0,\ a < 0\}$, even though in this case it is easy to deduce from the description above. Instead we only sample points from them and determine the number of solutions to the original system in a given region of parameter space.

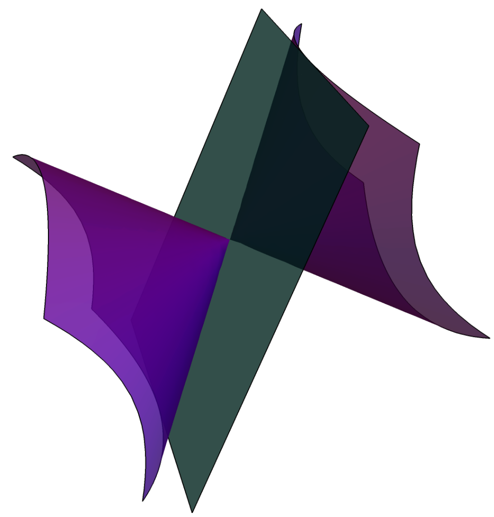

FIGURE 1. The surface $\mathbf{V}(b^2 - 4ac) \cup \mathbf{V}(a)$ bounding the connected components of M_3.

Next we consider the connected strata arising from $M_2 = \left(\mathbf{V}(b^2 - 4ac) \cup \mathbf{V}(a)\right) - \mathbf{V}(a, b)$, this again has 4 connected components, two arising from $\mathbf{V}(b^2 - 4ac) - \mathbf{V}(a, b)$, one of which has $c \geq 0$ and one $c \leq 0$, and two from $\mathbf{V}(a) - \mathbf{V}(a, b)$, one representative from each of these is: $(1, 2, 1) \in T_1$, $(-1, 2, -1) \in T_2$, $(0, 1, 1) \in T_3$, $(0, -1, 1) \in T_4$. We see that (3) has one real solution for all coefficients $(a, b, c) \in T_i$, for $i = 1, \ldots, 4$.

Next we consider the connected strata arising from $M_1 = \mathbf{V}(a, b) - \mathbf{V}(a, b, c)$. This has two connected components, one representative from each of these is: $(0, 0, 1) \in Z_1$, $(0, 0, -1) \in Z_2$ and (3) has no real solutions for all coefficients (a, b, c) in both Z_1 and Z_2.

Finally we have the closed stratum $\mathbf{V}(a, b, c)$ which is a single point and the corresponding system (3) has infinitely many solutions.

EXAMPLE 6.4. Consider the parametric system of equations in $\mathbb{R}^2$ given by:

$$x^2 - y^2 + b = -ax + x^2 + by = 0 \tag{4}$$

where we think of a, b as real parameters and x, y as real variables. To classify the solutions we consider the variety $X = \mathbf{V}(x^2 - y^2 + b, -ax + x^2 + by)$ in $\mathbb{R}^4$ and the projection map $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ onto $Y = \mathbb{R}^2$ defined by $(x, y, a, b) \mapsto (a, b)$. For the sake of brevity we display only the $Y_\bullet$ portion of the stratification of π as in Theorem 6.2, as for real root classification we in fact only use this part, it is:

$$Y_\bullet = \mathbb{R}^2 \supset \left(\mathbf{V}(a^6 - 3a^4b^2 + 3a^2b^4 - b^6 + a^4b - 20a^2b^3 - 8b^5 - 16b^4) \cup \mathbf{V}(b)\right) \supset \mathbf{V}(a,b) \cup \mathbf{V}(a,b+4).$$

The stratification is illustrated in Figure 2. Set

$$W = \mathbf{V}(a^6 - 3a^4b^2 + 3a^2b^4 - b^6 + a^4b - 20a^2b^3 - 8b^5 - 16b^4).$$

There are seven two dimensional connected strata making up $M_2 = \mathbb{R}^2 - (W \cup \mathbf{V}(b))$. A sample point in each of them is: $(0,2) \in S_1$, $(4,1) \in S_2$, $(-4,1) \in S_3$, $(0,-1) \in S_4$, $(0,-6) \in S_5$, $(-3,-4) \in S_6$, $(3,-4) \in S_7$. The system (4) has 4 real solutions for $(a,b) \in S_i$ for $i = 2,3,5$. The system (4) has 2 real solutions for $(a,b) \in S_i$ for $i = 1,6,7$. Finally the system (4) has no real solutions for $(a,b) \in S_4$.

There are eight one dimensional connected strata making up $M_1 = (W \cup \mathbf{V}(b)) - (\mathbf{V}(a,b) \cup \mathbf{V}(a,b+4))$. A sample point in each of them is: $(\frac{14}{27}, \frac{4}{27}) \in Z_1$, $(\frac{-14}{27}, \frac{4}{27}) \in Z_2$, $(-2,0) \in Z_3$, $(2,0) \in Z_4$, $(\frac{16}{27}, \frac{-16}{27}) \in Z_5$, $(\frac{-16}{27}, \frac{-16}{27}) \in Z_6$, $(2.74669..., -10) \in Z_7$, $(-2.74669..., -10) \in Z_8$. The system (4) has 3 real solutions for $(a,b) \in Z_i$ for $i = 1,2,3,4,7,8$. The system (4) has 1 real solution for $(a,b) \in Z_i$ for $i = 5,6$.

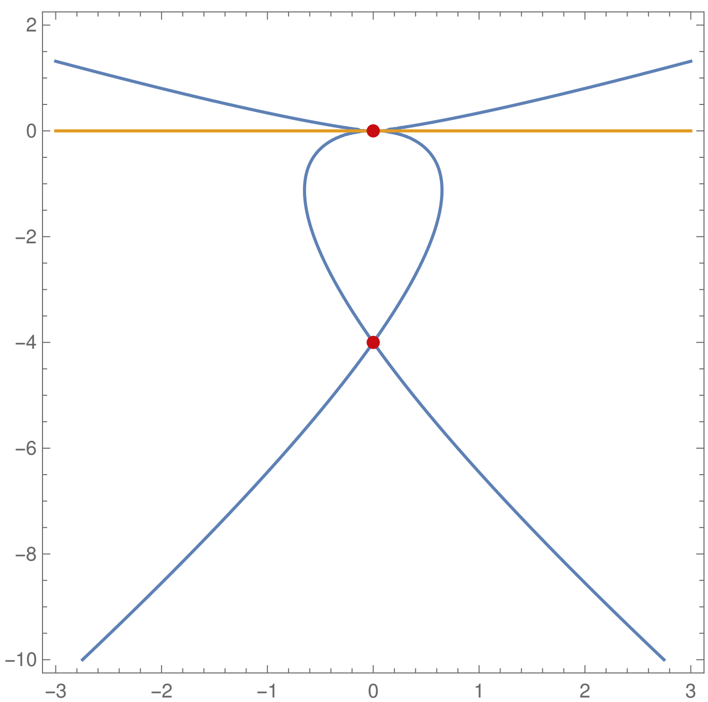

FIGURE 2. The algebraic constraints defining the connected strata of $Y_\bullet$.

Finally there are two zero dimensional strata, the points $(0,0), (0,-4)$. When $a = b = 0$ the system (4) has 1 real solution, when $a = 0, b = -4$ the system (4) has 2 real solutions.

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