GR7

Constants of the motion

(1) Show that if $B_a = \nabla_a f$ for some function f, then $\nabla_{[a}B_{b]} = 0$. The converse is also true: you may use this without proof.

Let F_{ab} be a solution of Maxwell's equations $\nabla_a F^{ab} = 0$, $\nabla_{[a}F_{bc]} = 0$ in curved space-time. The equation of motion of a particle of charge e and rest mass m is given by the Lorentz equation

$$mU^b\nabla_b U^a = eF^{ab}U_b,$$

where $U^a = \mathrm{d}x^a/\mathrm{d}s$, with s the proper time. Show that if the Lie derivative of F_{ab} along X^a vanishes (GR6, question 3), then $F_{ab}X^b = \nabla_a f$ for some function f. Show that if X^a is also a Killing vector then $mU^aX_a + ef$ is a constant of motion for the particle.

Orbits in the Schwarzschild space-time

(2) Show that along free particle worldlines in the equatorial plane of the Schwarzschild metric, the quantities

$$J = r^2\dot{\phi} \quad \text{and} \quad E = \left(1 - \frac{2m}{r}\right)\dot{t}$$

are constant (the dot is $\mathrm{d}/\mathrm{d}s$). Explain why the particle cannot escape to infinity if $E < 1$.

Show that

$$\dot{r}^2 + \left(1 + \frac{J^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = E^2,$$

$$\ddot{r} + \frac{m}{r^2} - \frac{J^2}{r^3} + 3\frac{mJ^2}{r^4} = 0.$$

For a circular orbit at radius $r = R$, show that

$$J^2 = \frac{mR^2}{R - 3m}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{m}{R^3}\right)^{1/2}.$$

Show by setting $r(s) = R + \epsilon(s)$, with ϵ small, that the circular orbit is stable if and only if $R > 6m$.

(3) Show that for a suitable value of $\alpha = mE/J$, there are equatorial null geodesics in the Schwarzschild solution on which

$$\frac{1 - 3u}{\left(\sqrt{3} + \sqrt{1 + 6u}\right)^2} = A\mathrm{e}^{\phi}$$

for arbitrary constant A. Describe their behaviour as $\phi \to -\infty$ for (i) $A > 0$ and (ii) $A < 0$. [You may quote $p^2 = \alpha^2 + 2u^3 - u^2$. Adapted from 1985 VIII 66.]