

Da). Introduction / Practical stuff

MAP
Bob
1/17/10

• Lecturer: Mason Porter

• website: <http://monlinea2010.blogspot.com>

• Lecturer: Mon : 12 - 1pm, MI, L1

Tues : 5 - 6pm, MI, L2

(1) " "

• classes:

(1) Tues : Me (3, 5, 7, 1)

TA : Martin Gould

Mon : 10:00 - 11:30am, DHSR3, weeks 3, 5, 7,

(2) Tutor : Me (3, 5, 7, 1)

TA : Martin Gould

Tues : 3 - 4:30pm, DHSR3, weeks 3, 5

11am - 12:30pm, DHSR3, week 7

(3) Tutor: Radetz Erban (3, 5, 7, 1)

TA: Puck Rombach

Tues : 1:30 - 3pm, DHSR3, week 3, 5

12:30 - 2pm, DHSR3, week 7

(4) Tutor: Radetz Erban (4, 5, 8, 1)

TA: Puck Rombach

Tues : 2:00 - 3:30pm, DHSR3, weeks 4, 6, 8

↳ box
will
originally
apply in
basement
of MI

→ • Problem Sheets for (1) - (4) are due Friday 5pm
of the previous week (#1 in Fri. of week 2,
etc.)

(4 problem sheet; #1 is posted as
well as scans of first 29 pages
of handwritten lecture notes; note: I am reworking
lecture notes & starting from scratch

(5) Tutor: Jasa Forbes (4, 6, 8, 11)

TA: Paul Moore

Fri: 9:30 - 11:00am, DHSR 2, weeks 4, 6, 8

(6) Tutor: Jasa Forbes

TA: Paul Moore

Fri: 2 - 3:30pm, DHSR 2, weeks 4, 6, 8

(7) Tutor: Kostas Zygalakis (3, 5, 7, 1)

TA: Amy Smith

Thurs: 10 - 11:30am, DHSR 2, weeks 3, 5, 7

(8) Tutor: Kostas Zygalakis

TA: Amy Smith

Thurs: 10 - 11:30am, DHSR 2, weeks 4, 6, 8

Problem sheets for (5) - (8) are due Monday of the same week.

- ④ I'll ask you to indicate preference tomorrow.
Come back w/ your top 4 (you must list ≥ 4)
& any strict conflict (not just inconvenience)
- I will have you fill this out tomorrow

Q, part 3)

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B28

- Nonlinear Science - my favorite subject!
(in fact, this is one of my specialties
& the original subject I was trained in)

- I hope to convey my excitement for the subject & show you some awesome things that can happen in nonlinear systems that simply can't occur in linear systems; I want to give a big picture overview of the field & motivate you to learn it further when the course is over

- awesome things include: • chaotic dynamics (of multiple sorts)

- You can find snippets of many of these in scholarpedia (online) & in Alwyn Scott's "Encyclopedia of Nonlinear Science"

- spontaneous emergence of order in very complicated systems (e.g., synchronization)

- propagation & interaction of complicated waves (e.g., solitons)

- etc. in the last couple lectures

I have to cover extra topics
according to student request
so let me know if you have any
requests!

- Nonlinear science is a highly visual subject

- while the course will have lots of analytical work, most research in nonlinear science is computational — accordingly, you will want to do computational experiments on your own as part of this course; this will help you immensely in learning this material!

- MATLAB (make sure you have this)

- lots of helpful free code available online (e.g., 'pydane' for planar dynamical systems)
(if you google, you can find MATLAB mfiles online)

0.) Preliminaries

website,

- classes, etc. (see results of organizational meeting) Matlab
 & tutor, TA (8 classes)

(download as it will be Matlab!)

1.) Bifurcations & Nonlinear Oscillators

- Books: see references in course synopsis; links to 2 online ones (Rand, (vitanovic') or course website)
- Nonlinear systems \approx Systems in which superposition doesn't hold (well, there are some exceptions - piecewise linear systems are effectively nonlinear in many respects even though superposition holds)

e.g., nonlinear dynamical system $\vec{\dot{x}} = \vec{f}(\vec{x})$, where some nonlinear function of x : ($\vec{x} = (x_1, \dots, x_n)$) appears somewhere on the RHS

l.e.s. / the pendulum ODE $\ddot{x} + \frac{g}{L} \sin x = 0$

gives $\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{L} \sin x \end{aligned}$ in dynamical system form

- one can also have nonlinear PDEs

(e.g., reaction-diffusion eqn) $\dot{x} = -\nabla^2 x + x^3$,
 nonlinear map ($x_{n+1} = x_n^2 + 2$), nonlinear delay eqn ($\dot{x} = f(x(t-\tau)) - [x(t)]^3$), etc.

haar! • as you'll see (& Matlab will help!), nonlinear systems exhibit a far richer class of behavior than linear ones

1b) 2-dim autonomous systems

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \quad (\text{no explicit time-dependence in RHS!})$$

recall)

- equilibrium points satisfy $\dot{x}_1 = \dot{x}_2 = 0$

↳ we can linearize near such points to examine stability

- suppose, (x_1^*, x_2^*) is an equilibrium $\Rightarrow f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$

↳ let $\begin{cases} u_1 = x_1 - x_1^* \\ u_2 = x_2 - x_2^* \end{cases}$ give a small disturbance (u_1, u_2) to the equilibrium

- plug into (1) to get:

$$\begin{aligned} \dot{u}_1 &= \dot{x}_1 = f_1(x_1^* + u_1, x_2^* + u_2) \\ &= f_1(x_1^*, x_2^*) + u_1 \frac{\partial f_1}{\partial x_1}(x_1^*, x_2^*) + u_2 \frac{\partial f_1}{\partial x_2}(x_1^*, x_2^*) + O(u_1^2, u_2^2, u_1 u_2) \\ &\stackrel{\text{Taylor}}{\rightarrow} 0 \\ \dot{u}_2 &= u_1 \frac{\partial f_2}{\partial x_1} + u_2 \frac{\partial f_2}{\partial x_2} + O(u_1^2, u_2^2, u_1 u_2) \end{aligned}$$

quadratic terms &
 above
 small

- similarly, $\dot{u}_2 = u_1 \frac{\partial f_2}{\partial x_1} + u_2 \frac{\partial f_2}{\partial x_2} + O(u_1^2, u_2^2, u_1 u_2)$

∴ the linearized system is

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}}_{\text{Jacobian } J} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

"linear stability"
 (recall) (x_1^*, x_2^*)

- Stability: near any equilibrium, you compute the eigenvalues of $J(x_1^*, x_2^*)$ & examine their signs (possibly as a function of system parameters)

↳ review notes from Part A to recall

2a]

Conservative Systems

MAT
B2b
nonlinear 5
1/12/10

- Consider a particle of mass m that moves along x -axis with nonlinear restoring force $F(x)$

equations of motion: $m\ddot{x} = F(x)$

[assuming F is indep. of \dot{x}, t]

↳ no damping or friction of any kind, no time-dep. in driving force

↳ energy is conserved; letting $F = -\frac{dV}{dx}$

(V is potential energy), you can show that $\dot{E} = 0$, where $E = \frac{1}{2}mv^2 + V(x)$

- More generally (except in a case called "non-holonomic" - think of bicycles - which we won't study), this is an example of a Hamiltonian system $H(q, p) = \text{const.}$, whose

eqn. of motion are $\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$ (2)

- Such systems cannot have attracting equilibria (show this at home)

e.g.) Duffing oscillator

- particle of mass m moving in a double well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$; let's find & classify all equilibria & then plot the phase portrait

2b] $\cdot F(x) = -\frac{dV}{dx} = x - x^3$, so $\ddot{x} = x - x^3 \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$ (3)

• equilibria at $\dot{x} = \dot{y} = 0 \Rightarrow (0,0), (\pm 1, 0)$

• $J = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$

(eigenvalues)

• $J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det |J - I\lambda| = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$

$$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

\Rightarrow saddle (unstable)

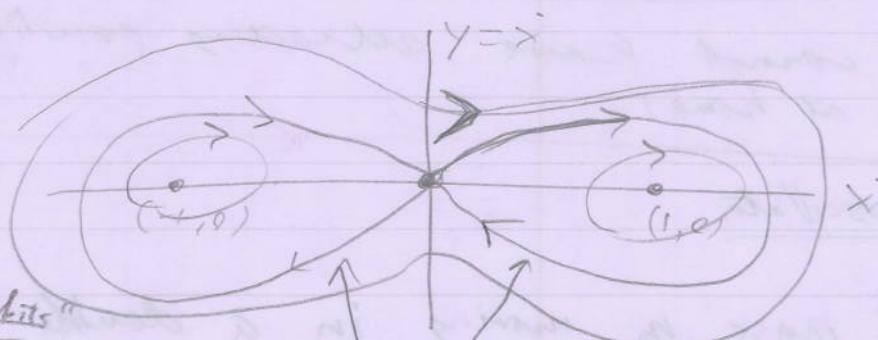
• $J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow \det |J - I\lambda| = \det \begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda \end{pmatrix} = 0$

$$\Rightarrow \lambda^2 + 2 = 0 \Rightarrow \lambda = \pm i\sqrt{2}$$

\Rightarrow center

• The center could be destroyed in general by non-linear perturbations but the conservative nature of (3) prevents it here

• in fact, our phase portrait is described by orbits given by $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{const.}$



$(\dot{x} > 0 \text{ for } y > 0$
gives direction
of arrows)

"heteroclinic orbits"
when 2 different equilib.
are connected
(rare)
common in conservative
systems because of
symmetry; usually

homoclinic orbits (connect an equilib.
to itself)
("stable manifold" connects to)
("unstable manifold")

approach origin
 $\text{as } t \rightarrow \pm\infty$

3a) Thm. (nonlinear center for conservative systems)

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- Consider $\dot{\vec{x}} = \vec{f}(\vec{x})$, where $\vec{x} = (x, y) \in \mathbb{R}^2$ & \vec{f} is continuously differentiable; suppose \exists a conserved quantity $E(\vec{x})$ & that \vec{x}^* is an isolated equilibrium (i.e., no other equilibria in a neighborhood surrounding \vec{x}^*). If \vec{x}^* is a local minimum of E , then all trajectories sufficiently close to \vec{x}^* are closed.
- Proof: Trajectories close at home & see why if desired
- Remark: \exists a similar result for reversible systems
 (e.g., $\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$ s.t. $f(x, -y) = -f(x, y)$ $g(x, -y) = g(x, y)$)
 (this is invariant under $(x, y) \mapsto (-x, -y)$,
 $m\ddot{x} = F(x)$ is an example of something of this form)

• 1 Non-conservative oscillations

- e.g., pendulum with friction, Lienard systems (e.g., van der Pol), typical biological oscillators, etc.
- e.g., Van der Pol

def: a limit cycle is an isolated closed trajectory

(the closed orbits we saw in conservative systems were non-isolated; symmetry imposed a whole family of them)

- They can be stable, unstable, or half stable



stable



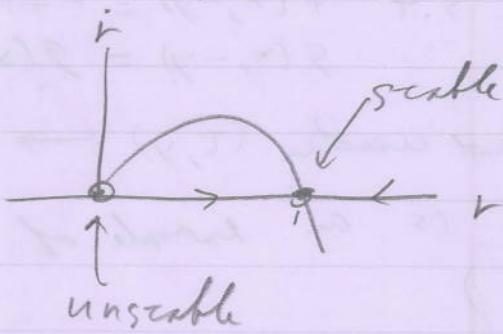
unstable



half stable
(not always called this)

$$\text{e.g.) } \begin{cases} \dot{r} = r(1-r^2) & (5) \\ \dot{\theta} = 1 \end{cases}$$

- can analyze as a 1D dynamical system in r
- $\dot{r} = 0 \Rightarrow r^* = 0, r^* = 1$ ($r \geq 0$ because $r = \text{radius}$)
(equilibria)



e.g.) van der Pol oscillator

- $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, $\mu \geq 0$ is a parameter
(can have different types of dynamics for different values of μ)

- arose historically in connection w/ nonlinear electric used in first radios (similar eqns now show up in places like mathematical biology)
- $\mu = 0 \Rightarrow$ simple harmonic oscillator

4a)

- however, we have nonlinear damping:
- $u(x^2 - 1)\dot{x}$ gives positive damping for $|x| > 1$
& negative damping for $|x| < 1$; this balances to produce a self-sustained oscillation (a limit cycle)
- sometimes one can rule out closed orbits
 - e.g.) gradient systems, $\dot{\vec{x}} = -\nabla V$ for some potential don't have closed orbits
(sketch of proof: suppose \exists one; see what happens after one circuit & obtain contradiction)
 - e.g.) Liapunov function; consider $\dot{\vec{x}} = \vec{f}(\vec{x})$ with equilibrium at \vec{x}^* : if we find a "Liapunov function" $V(\vec{x})$ (which is real-valued, C^1 ($=$ continuously differentiable)) s.t.
 - i.e. V is "positive definite"
 - $\begin{cases} (1) V(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{x}^*, \quad V(\vec{x}^*) = 0 \\ (2) \dot{V} < 0 \quad \forall \vec{x} \neq \vec{x}^*, \end{cases}$
 Then \vec{x}^* is globally asymptotically stable (all trajectories approach it as $t \rightarrow \infty$), so the system can't have closed orbits

\vec{x}^* because V always strictly negative & cst at \vec{x}^*
- Dulac's criterion: let $\dot{\vec{x}} = \vec{f}(\vec{x})$ be C^1 & defined on a simply connected region $R \subset \mathbb{R}^2$; if \exists a C^1 real-valued $g(\vec{x})$ s.t. $\vec{D} \cdot (g \vec{f})$ has one sign throughout R , then \nexists closed orbits lying entirely in R

• Poincaré-Bendixson Thm. (note: can be used to show that 2D dynamical systems can't be chaotic)

• Thm [P-B]: Suppose that:

(1) $R \subset \mathbb{R}^2$ is closed & bounded

(2) $\dot{\vec{x}} = \vec{f}(\vec{x})$ is C^1 on an open set containing R

(3) R does not contain equilibria

(4) \exists a trajectory C that is confined in R (it starts in R & stays there $\forall t > 0$)

Then either C is a closed orbit or it spirals towards a closed orbit as $t \rightarrow \infty$

(note: $t \mapsto -t$ for, $\forall t < 0$ for unstable case)

↓ (4) is the tough condition to satisfy $\xrightarrow[(1)-(3) \text{ are easy to check}]{} (1)-(3)$

[for proof, see, e.g., the book by Peitro]

↳ this is the rigorous version of the closed orbit getting trapped in the region

$$\text{e.g. } \begin{cases} \dot{r} = r(1-r^2) + \mu r \cos \theta \\ \dot{\theta} = 1 \end{cases} \quad (6)$$

• $\mu = 0 \Rightarrow$ stable limit cycle at $r=1$ (see earlier example)
 we'll show that there's still a closed orbit for sufficiently small $\mu > 0$

• we seek concentric circles w/ radii r_{\min} & r_{\max} s.t.
 $\dot{r} < 0$ on r_{\max} & $\dot{r} > 0$ on r_{\min}

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- Then the annulus $0 < r_{\min} \leq r \leq r_{\max}$ is our desired "trapping region" R
- $\dot{\theta} > 0 \Rightarrow$ no equilibria in R , so P-B guarantees a stable closed orbit if we find r_{\min} & r_{\max}
- to find r_{\min} , we require $\dot{r} = r(1 - r^2) + \mu r \cos \theta > 0 \forall \theta$
 - $\cos \theta \geq -1$, so $r_{\min} = 1 - r^2 - \mu > 0$ works as long as $r_{\min} < \sqrt{1 - \mu}$
(in fact, closed orbits can't exist for $\mu \geq 1$, but we can't use this argument to set them; must closed orbits exist $\forall \mu > 0$?)
- ($r_{\min} = \sqrt{1 - \mu}$ will actually work, but more careful reasoning would then be required)
- similarly, we need $r_{\max} > \sqrt{1 - \mu} + \mu$

Liénard Systems (generalization of VdP)

- $\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (7)$ "Liénard eqn."
- mechanical interpretation (though this isn't how the eqn is used): eqn. of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ & a restoring force $-g(x)$
- can be written as $\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - f(x)y \end{cases} \quad (8)$

• Lienard's Thm: Suppose $f(x)$ & $g(x)$ satisfy:

(1) $f(x), g(x)$ are C^1

(2) $g(-x) = -g(x) \quad \forall x$

(3) $g(x) > 0$ for $x > 0$

(4) $f(-x) = f(x) \quad \forall x$

(5) the (odd) function $F(x) = \int_0^x f(u) du$ has exactly one positive zero at $x=a$, is negative for $0 < x < a$, is positive & nondecreasing for $x > a$, & $F(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Then (8) has a unique, stable limit cycle surrounding the origin.

• (2,3) \Rightarrow restoring force acts like ordinary spring & reduces displacement

• assumptions on $f \Rightarrow$ negative damping for small $|x|$ & positive damping for large $|x|$

• for a proof, see Jordan & Smith or Perko

e.g.) Show that VdP has a unique, stable limit cycle

• $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has $f(x) = \mu(x^2 - 1)$, $g(x) = x$,
so $\stackrel{\text{conditions}}{\nexists}$ (1) - (4) are satisfied

• to check (5), observe $F(x) = \mu\left(\frac{1}{3}x^3 - x\right) = \frac{1}{3}\mu x(x^2 - 3)$,
so we satisfy (5) w/ $a = \sqrt{3}$

\therefore VdP has a unique stable limit cycle

Relaxation Oscillations

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BSB

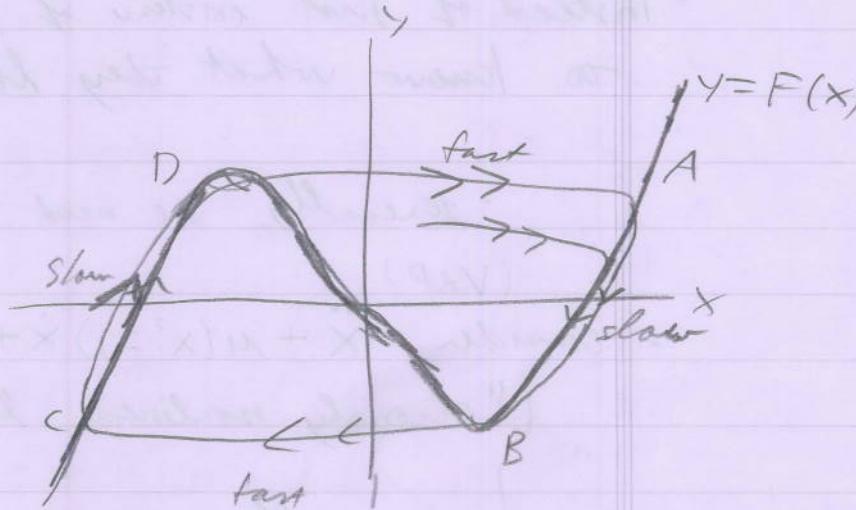
- instead of just existence of periodic sol, we really want to know what they look like
 - generally, we need to be approximate (VdP)
- consider $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, $\mu \gg 1$
 ("strongly nonlinear limit")
 - ↳ as well see, the limit cycle consists of an extremely slow build-up followed by a sudden discharge (rinse, wash, repeat); such oscillations are called relaxation oscillations (the 'stress' accumulated during build-up is 'relaxed' during discharge)
 - relaxation oscillations occur, e.g., in periodic firing of nerve cells driven by const current
 - let's do a phase plane analysis:
 - ↳ observe $\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}(\dot{x} + \mu[\frac{1}{3}x^3 - x])$
 - let $w = \dot{x} + \mu F(x)$ s.t. $\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x$
 - $\Rightarrow \begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$ (9)
 - letting $y = \frac{w}{\mu}$ then given $\begin{cases} \dot{x} = \mu[y - F(x)] \\ \dot{y} = -\frac{1}{\mu}x \end{cases}$ (10)

Now consider a typical trajectory in the (x,y) -plane

↳ Use nullclines ($\dot{x}=0$ gives a set; vertical motion)
 $\dot{y}=0$ gives another; horizontal)
 to understand
 motion, which I claim
 is as follows:

- starting from anywhere except $(0,0)$, the trajectory goes horizontally onto the cubic nullcline

$$y = F(x) \quad \left\{ \begin{array}{l} \text{from } \dot{x} = 0 \end{array} \right\} \text{ at } A \text{ or } C$$



- then it goes slowly along the nullcline until reaching a "knee" (B & D); & then it jumps horizontally to the other branch ((or A))
- To see this, suppose that the initial condition is not too close to cubic nullcline; i.e., $y - F(x) \sim O(1)$

$$\hookrightarrow \text{then (9)} \Rightarrow \begin{cases} |\dot{x}| \sim O(u) \gg 1 \\ |\dot{y}| \sim O(u') \ll 1 \end{cases} \quad (\text{2 timescales})$$

\Rightarrow enormous horizontal velocity,
 tiny vertical velocity

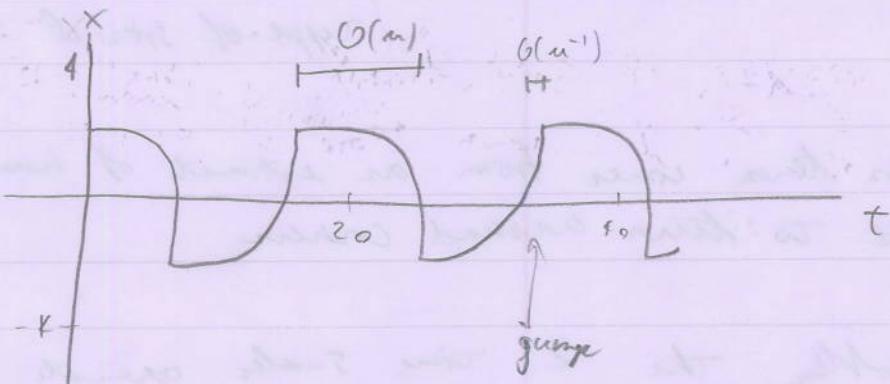
\Rightarrow trajectory is almost horizontal

i.f. i.e. is above cubic nullcline, then $y - F(x) > 0 \Rightarrow \dot{x} >$
 \Rightarrow jump to right (jump to right if below)

- once trajectory gets close enough s.t.

$y - F(x) \sim O(n^{-2})$, then $\dot{x} \sim \dot{y} \sim O(n^{-1})$, so the trajectory crosses the nullcline vertically & moves slowly along its backside w/ a velocity of size $O(n^{-1})$ until it reaches a knee & jumps sideways again

- we can see the timescales in the sol. $x(t)$



- now, let's estimate the period of the oscillation:

- the period T is essentially the time required to travel along the 2 slow branches (because time spent doing jumps is negligible for large n)

- by symmetry, the time on each branch is the same, so $T \approx 2 \int_{t_1}^{t_2} dt$

- on slow branches, $y \approx F(x) \Rightarrow \frac{dy}{dt} = F'(x) \frac{dx}{dt} = (x^2 - 1) \frac{dx}{dt}$

- using $\frac{dy}{du} = -\frac{x}{n}$, we get $\frac{dx}{dt} = -\frac{x}{n}(x^2 - 1)$

$$\Rightarrow dt = \frac{n(x^2 - 1)}{x} dx \text{ on a slow branch}$$

- the positive branch begins at $x_A = 2$ & ends at $x_B = 1$
(check this!)

$$\Rightarrow T = 2 \int_2^1 \frac{1}{x} (x^2 - 1) dx = 2 \mu \left[\frac{x^2}{2} - \ln x \right]_2^1 = \mu (3 - 2 \ln 2),$$

which is $O(\mu)$ as expected
(check this w/
Matlab!)

- with more work, one can show that

$$T = \mu [3 - 2 \ln 2] + 2 \alpha \mu^{-1/3}, \text{ where } \alpha \approx 2.338 \text{ is the smallest root of } A_i(-\alpha) = 0. \quad \begin{cases} A_i(x) = \text{Airy function, a type of special function} \end{cases}$$

- convection term comes from an estimate of how long it takes to turn around corners

- in this problem, the 2 time scales operate sequentially.
We're about to see stuff where they operate concurrently

weakly nonlinear oscillators

- they have the form $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad (11)$
 $0 \leq \epsilon \ll 1$, h is a smooth function

e.g., appropriate $h(x, \dot{x})$ give weakly nonlinear versions of Duffing ($h(x, \dot{x}) = x^3$) & VdP ($h(x, \dot{x}) = (x^2 - 1)\dot{x}$) oscillators

- we'll use weakly nonlinear oscillators to look at situations with 2 time scales simultaneously
(which occurs typically in real situations)

8a]

- Two time scale perturbation theory (much more on this in C6.3a)

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B8b

- Let $\begin{cases} T = \tau & \text{denote O(1) time} \\ T = \epsilon\tau & \text{denote slow time (recall: } |\epsilon| \text{ small)} \end{cases}$
- functions of T will be treated as constants on the fast time scale τ
- expand the solution of (11) in a series

$$x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2) \quad (12)$$
- use the chain rule to transform time derivatives:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial t} \frac{dt}{d\tau} + \frac{\partial x}{\partial T} \frac{dT}{d\tau} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T}$$
- using expansion (12), we get:

$$\dot{x} = \frac{\partial x_0}{\partial \tau} + \epsilon \left(\frac{\partial x_0}{\partial T} + \frac{\partial x_1}{\partial \tau} \right) + O(\epsilon^2) \quad (13)$$
- similarly, $\ddot{x} = \frac{\partial^2 x_0}{\partial \tau^2} + \epsilon \left(\frac{\partial^2 x_1}{\partial T^2} + 2 \frac{\partial x_0}{\partial \tau \partial T} \right) + O(\epsilon^2) \quad (14)$

e.g.) we'll use this to show that VdP oscillator (weak nonlinearity) has a nearly-circular limit cycle with radius $= 2 + O(\epsilon)$ and frequency $\omega = 1 + O(\epsilon^2)$

- $\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0 \quad (\text{small } \epsilon)$
- use (13, 14) & collect small powers of ϵ , which we'll solve one at a time:

- $O(1): \frac{d^2x_0}{d\tau^2} + x_0 = 0$

- $O(\epsilon): \frac{d^2x_1}{d\tau^2} + x_1 = -2\frac{d^2x_0}{dt d\tau} - (x_0^2 - 1)\frac{dx_0}{d\tau} \quad (15)$

- for weakly nonlinear oscillator, the $O(1)$ eqn is always a simple harmonic oscillator

↳ can write its general sol. as $x_0 = r(\tau) \cos(\tau + \phi(\tau))$
 (τ is treated as const. when integrating on fast timescale)

• $r(\tau), \phi(\tau)$ are slowly-varying amplitude & phase

- insert (16) into (15) to find eqns for $r(\tau), \phi(\tau)$:

$$\Rightarrow \frac{d^2x_1}{d\tau^2} + x_1 = -2 \left(\frac{dr}{d\tau} \sin(\tau + \phi) + r \frac{d\phi}{d\tau} \cos(\tau + \phi) \right) - r \sin(\tau + \phi) [r^2 \omega^2 (\tau + \phi) - 1] \quad (17)$$

↗ (or "secular")

- We want a bounded sol, so we can't have "resonant" terms on RHS (these are trig terms of same frequency as approx x_0)

⇒ the overall coeff of $\sin(\tau + \phi), \cos(\tau + \phi)$ must be 0

- expand RHS of (17) with trig identity:

$$\sin(\tau + \phi) \cos^2(\tau + \phi) = \frac{1}{4} [\sin(\tau + \phi) + \sin 3(\tau + \phi)]$$

$$\Rightarrow \frac{d^2x_1}{d\tau^2} + x_1 = \left[-2 \frac{dr}{d\tau} + r - \frac{1}{4} r^3 \right] \sin(\tau + \phi) + \left[-2r\phi' \right] \cos(\tau + \phi) - \frac{1}{4} r^3 \sin 3(\tau + \phi)$$

↗ "slow flow equation"

- to avoid secular terms, we thus require:

$$\begin{cases} -2 \frac{dr}{d\tau} + r - \frac{1}{4} r^3 = 0 \\ -2r\phi' = 0 \end{cases}$$

(18)

of (18)

- the 1st eqn. can be written as

$$r' = \frac{1}{\epsilon} r(4 - r^2) \quad (19)$$

(a 1D dynamical system on half-line $r \geq 0$)

- can show that $r=0$ is unstable equilibrium
 $\Delta r=2$ is stable equilib (here, the Jacobian is a scalar — a 1×1 matrix of the first partial derivatives)

$$\Rightarrow r(T) \rightarrow 2 \text{ as } T \rightarrow \infty$$

- 2nd eqn of (18) $\Rightarrow \dot{\phi}' = 0 \Rightarrow \phi(T) = \phi_0 = \text{const}$

$$\therefore x_0(\tau, T) \rightarrow 2 \cos(\tau + \phi_0), \text{ so}$$

$$x(\tau) \rightarrow 2 \cos(\tau + \phi_0) + O(\epsilon) \text{ as } \tau \rightarrow \infty, \text{ so}$$

$x(\tau) \rightarrow$ a stable limit cycle of radius $2 + O(\epsilon)$

(angular)

- To find the frequency of oscillations, let $\theta = \tau + \phi(\tau)$ & the angular freq. is

$$\omega = \frac{d\theta}{d\tau} = 1 + \frac{d\phi}{dT} \frac{dT}{d\tau} = 1 + \epsilon \phi' = 1 + O(\epsilon^2)$$

- To find the $O(\epsilon^2)$ correction, we'd introduce a super slow time scale $T_s = \epsilon^2 t$ or use the "Poincaré-Lindstedt method"
- "averaging" (see exercises)

9b) Introduction to Bifurcations

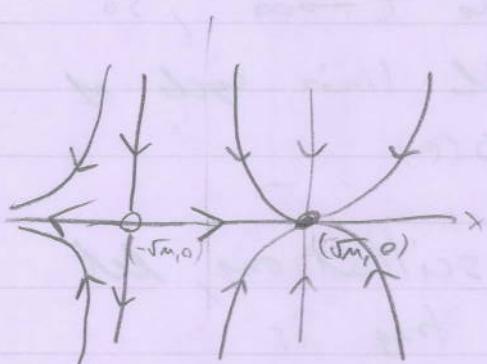
- if a phase portrait changes its topological structure as one or more parameters are varied, a bifurcation is said to have occurred (imagine controlling a parameter w/ a knob in an experiment)
- among other things, we can use this to start describing ways in which oscillations can be turned on and off ("codimension" $\stackrel{\text{mind} \#}{=}$ of parameters that are varied so bif to occur)
- saddle-node bifurcation (also called 'fold')

- basic mechanism for creation and destruction of equilibria

all interesting dynamics occur here

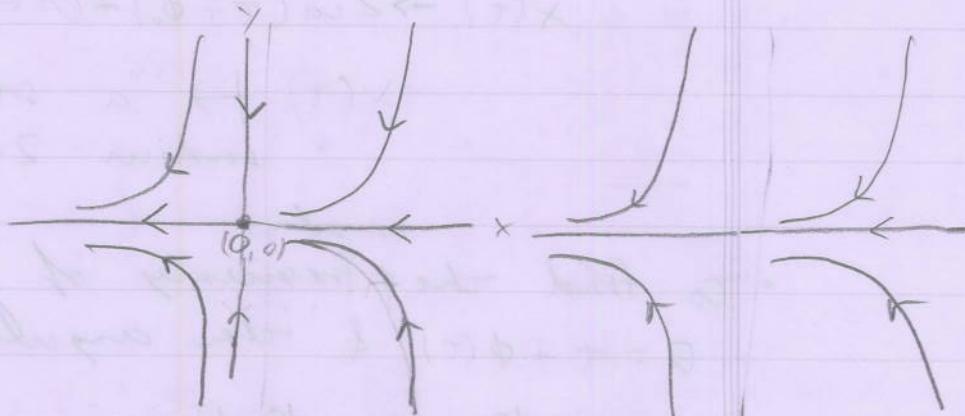
• prototypical example: $\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases}$ (20)

exponentially damped



$$\mu > 0$$

(saddle + node)



$$\mu = 0$$

$$\mu < 0$$

(no equilib.)

When I asked about bifurcations,
I expect the bif. point to be considered
as well (not just before & after)

10a

e.g.) (model for a genetic control system;
Griffith 1971)

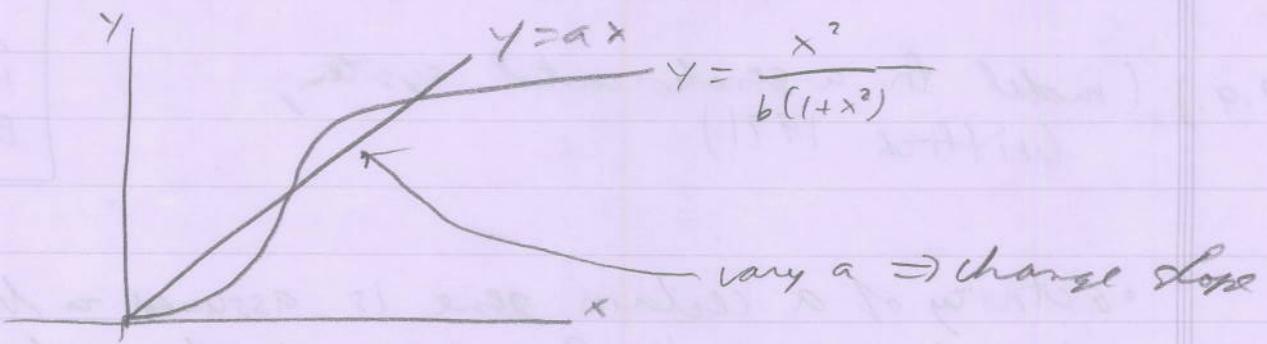
MAP
11/14/10
B8b

- activity of a certain gene is assumed to be directly induced by 2 copies of the protein for which it codes (i.e., gene is stimulated by its own product, potentially leading to an autocatalytic feedback process)
- dimensionless equations:

$$\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \frac{x^2}{1+x^2} - by, \end{cases} \quad (2.1)$$

where: x & concentration of protein
 y & concentration of messenger RNA
 a, b parameters that govern the degradation rates of x, y

- will show that there are equilibria for $a < a_c$ (will determine a_c) & that there is a saddle-node bifurcation at $a = a_c$
- nullclines: $\dot{x} = 0 \Rightarrow y = ax$
 $\dot{y} = 0 \Rightarrow y = \frac{x^2}{b(1+x^2)}$ ("sigmoidal curve") often shows up in biological problems
- let's vary a while holding b fixed



- small $a \Rightarrow 3$ intersections ($\Rightarrow 3$ equilibria)

- top 2 intersections coalesce as a increases until only $(0,0)$ is an equilibrium

- equilibria are $(0,0)$ or $ab(1+x^2) = x^*$

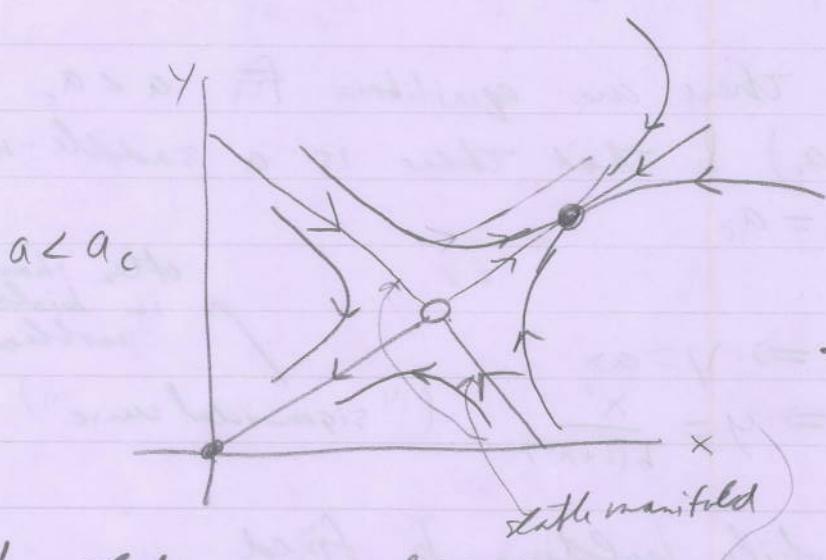
$$\Rightarrow x^* = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}$$

$(2ab < 1 \text{ for } 2 \text{ solutions})$

- coalescence at $2ab = 1 \Rightarrow a_c = \frac{1}{2b}$ ($\& x^* = 1$ at the bifurcation)

- Taubian $J(x,y) = \begin{pmatrix} -a & 1 \\ 2x & \frac{-b}{(1+x^2)^2} \end{pmatrix}$, & you can compute and

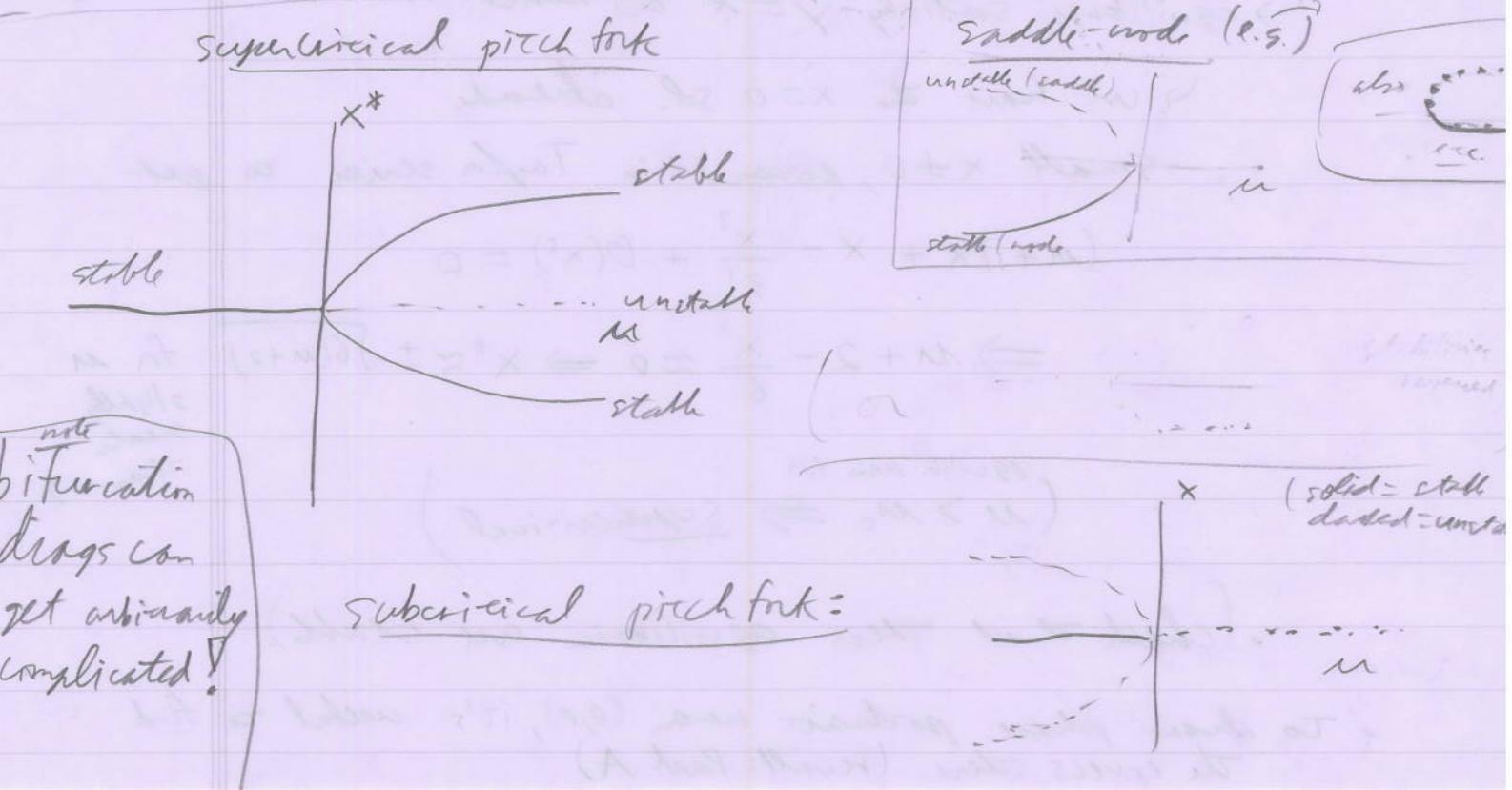
the smaller root $x^* \in (0,1)$ is a saddle & that the larger root $x^* > 1$ is a stable node
 $(0,0)$ is always a stable node



The stable manifold of the saddle acts like a threshold; it determines whether the gene turns on or off (depending on initial (x,y))

- biological interpretation: the system can act like a biochemical switch but only if mRNA & protein decay slowly enough ($ab < \frac{1}{2}$). There are then 2 stable steady states

- 11(a) Transcritical & pitchfork bifurcations
- MAP
4/14/10
B8b
- prototypical examples (btw, there are more, conceptually called "normal forms")
 - $\begin{cases} \dot{x} = mx - x^2 \\ \dot{y} = \dot{y} = y \end{cases}$ (transcritical)
 - $\begin{cases} \dot{x} = mx - x^3 \\ \dot{y} = -y \end{cases}$ ("forward bif" (in other place like bio))
you can look up when the norm. datum comes down on your own ($=$ "backward bif")
 - $\begin{cases} \dot{x} = mx + x^3 \\ \dot{y} = -y \end{cases}$ (subcritical pitchfork)
location of equilibrium pt.
-
- "You get a switch in stability"
- So this is a 1D phenomenon when in normal form



$$\text{e.g.) } \begin{cases} \dot{x} = ux + y + \sin x \\ \dot{y} = x - y \end{cases} \quad (22)$$

- (22) is invariant under change of var $x \rightarrow -x, y \rightarrow -y$
so the phase portrait is symmetric under reflection through $(0,0)$

- $(0,0)$ is an equilibrium $\forall u$ & $J(0,0) = \begin{pmatrix} u+1 & 1 \\ 1 & -1 \end{pmatrix}$
(compute eigenvals as usual)
 - ↪ it is stable (node) for $u < -2$ & a saddle (unstable) for $u > -2$
 - ↪ this suggests that a pitchfork bif. occurs at $u_c = -2$

- To continue this, we seek a symmetric pair of equilibria near the origin for $u \approx u_c$.

↪ equilibria satisfy $y = x$ & hence $(u+1)x + \sin x = 0$

↪ we have the $x=0$ sol. already

- Small $x \neq 0$; expand in Taylor series to get

$$(u+1)x + x - \frac{x^3}{3!} + O(x^5) = 0$$

$$\Rightarrow u+2 - \frac{x^2}{6} \approx 0 \Rightarrow x^* \approx \pm \sqrt{6(u+2)}$$

(equilib. pair for
 $u > u_c \Rightarrow$ supercritical)

for u
slightly
greater
than -2

- Check that these equilibria are stable
- To draw phase portrait near $(0,0)$, it's useful to find the eigenvs there (recall: Part A)

12a]

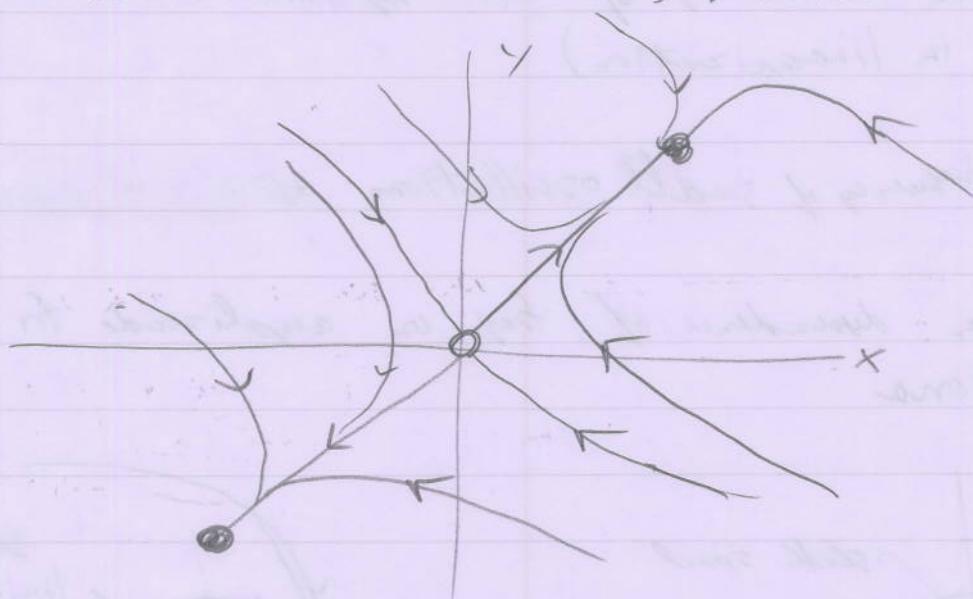
 MAP
 1/14/10
 B8b

- one can do it exactly, but we can get a good approx (a faster calculation!)

by noting that $\left. J \right|_{n=M_c} \approx \left. J \right|_{n=M_c=-2}$

$$\Rightarrow J \approx \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ which has e-vecs } \begin{pmatrix} 1 & 1 \end{pmatrix} \quad (\lambda=0) \\ \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (\lambda=-2)$$

- for n slightly greater than M_c , we have a saddle so the \mathcal{O} -val is slightly positive



Hopf bifurcations

- suppose a 2D dynamical system has a stable equilib as a param n is varied
- ↳ the ways stability can be lost are governed by the eigenvalues of J (all $\text{Re}(\lambda) < 0 \Rightarrow$ stable; any $\text{Re}(\lambda) > 0 \Rightarrow$ unstable)
- also, if $\lambda \notin \mathbb{R}$, we get oscillations

- in Hopf bifurcations, 2 cplx conj. λ simultaneously cross imaginary axis

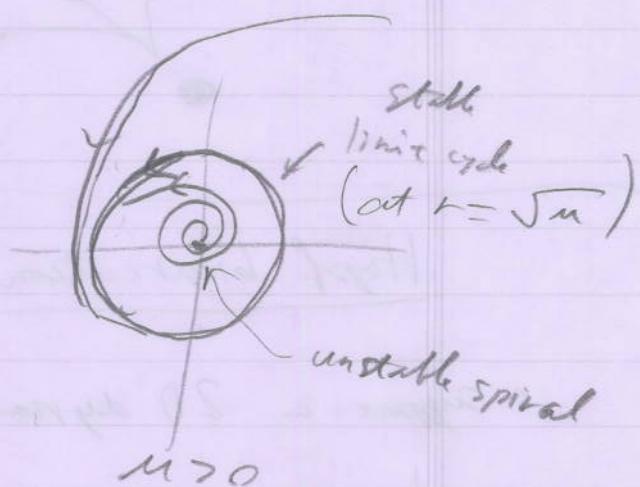
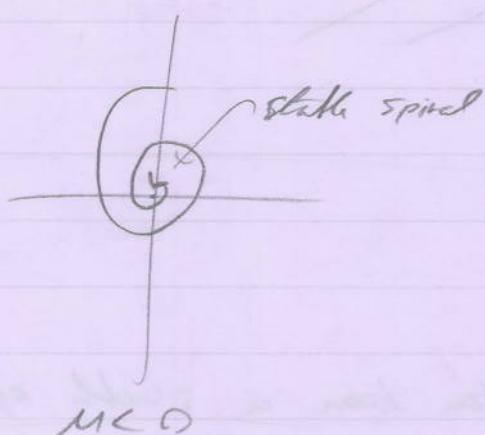
- supercritical Hopf: $\mu < \mu_c$: stable equilibrium (spiral)
 $\mu > \mu_c$: unstable equilibrium but we have a stable limit cycle that was also created (small-amp for $\mu \approx \mu_c$)

e.g.) $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases} \quad (23)$

μ controls stability of $(0,0)$ equilib. (all other params vanish in linearization)

ω gives frequency of small oscillations about the point

b determines dependence of freq. on amplitude for larger oscillations



Let's look at the e-vale:

need to convert $\stackrel{(23)}{\rightarrow}$ to Cartesian coordinates

$$\dot{x} = \mu x - wy + \text{cubic terms}$$

$$\dot{y} = wx + my + \text{cubic terms}$$

MAP
1/14/10
BD

$$\Rightarrow J(0,0) = \begin{pmatrix} m - \omega & \\ \omega & m \end{pmatrix}$$

$\Rightarrow \lambda = m + i\omega$, so the eigenvalues cross the imaginary axis left to right as m increases from < 0 to > 0

(generic situation)

- rule of thumb (but of course there are exceptions...)
 - the size of the limit cycle grows continuously from zero & increase proportionally to $\sqrt{m-m_c}$ for $m \approx m_c$
 - the freq. of the limit cycle is given approximately by $\omega = \text{Im } \lambda$ (evaluated at $m=m_c$)
 - exact at birth of limit cycle & correct up to $O(m-m_c)$ for $m \approx m_c$; the period is then $T = \left(\frac{2\pi}{\text{Im } \lambda}\right) + O(m-m_c)$
 - note that there are also subcritical & degenerate Hopf bifurcations

Global Bifurcations of Cycles

- other ways to create & destroy limit cycles (Hopf bifurcation is local in that it arises through change at an equilib. pt.)
- global bifurcations, by contrast, involve regions of phase space beyond just the neighborhood of a single equilib.

Saddle-Node Bifurcation of Cycles

- just like regular saddle-node but for limit cycles

e.g., $\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}, \mu < 0 \quad (24)$

- regard the radial eqn as a 1D dynamical system
 - check: this system undergoes a saddle-node bif at $\mu_c = -1/4$
 - in the 2D system, such pts. correspond to limit cycles



- an $O(1)$ -amp cycle is born out of nowhere (by contrast, for a Hopf bif, it grows proportionally to $\sqrt{\mu-\mu_c}$)

14a

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 B86

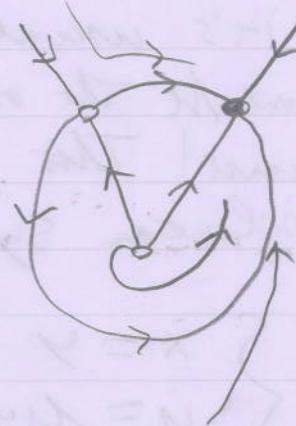
Infinite-Period Bif

e.g.) $\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = \mu - \sin \theta \end{cases}, \mu \geq 0 \quad (25)$

- note: r & θ are decoupled, so this is actually just two 1D systems

r equals all trajectories (except equilib $r^* = 0$) approach circle $r=1$ monotonically as $t \rightarrow \infty$ everywhere

θ eqn: counterclockwise motion for $\mu > 1$ whereas there are two invariant rays (defined by $\sin \theta = \mu$) if $\mu < 1$

 $\mu > 1$  $\mu < 1$

- as μ decreases, the limit cycle $r=1$ develops a bottleneck at $\frac{\partial}{\partial^2}$ that becomes increasingly severe as $\mu \rightarrow 1^+$; the oscillation period lengthens & finally becomes infinite at $\mu_c = 1$, when an equilib appears on the circle. for $\mu < \mu_c$ the point splits into a saddle & a node

• as the bif. is approached, the oscillation amp. stays $O(1)$ but the period increases as $(M - M_c)^{-\frac{1}{2}}$

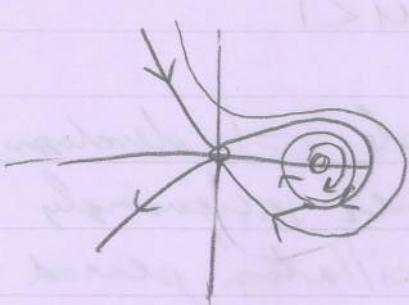
mechanism

• note: we can see this in generally coupled 2D examples

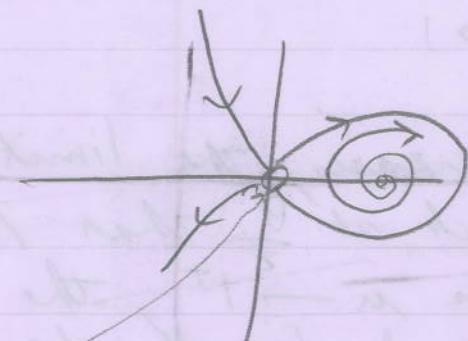
Homoclinic Bits

- in this scenario, a limit cycle moves progressively closer to a saddle; at the bif., the cycle touches the saddle & becomes a homoclinic orbit (this is another type of infinite-period bif.)
- hard to find analytically tractable example, so it's usually done computationally (this won't be the only time that happens in this course! This is in fact a feature of nonlinear systems!)

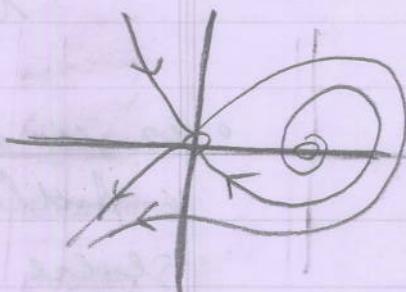
$$\text{e.g. } \begin{cases} \dot{x} = y \\ \dot{y} = My + x - x^2 + xy \end{cases} \quad (26)$$



$$M = -0.92 \quad (M < M_c)$$



$$M = M_c \approx -0.8695 \quad (\text{now connects at this pt.})$$



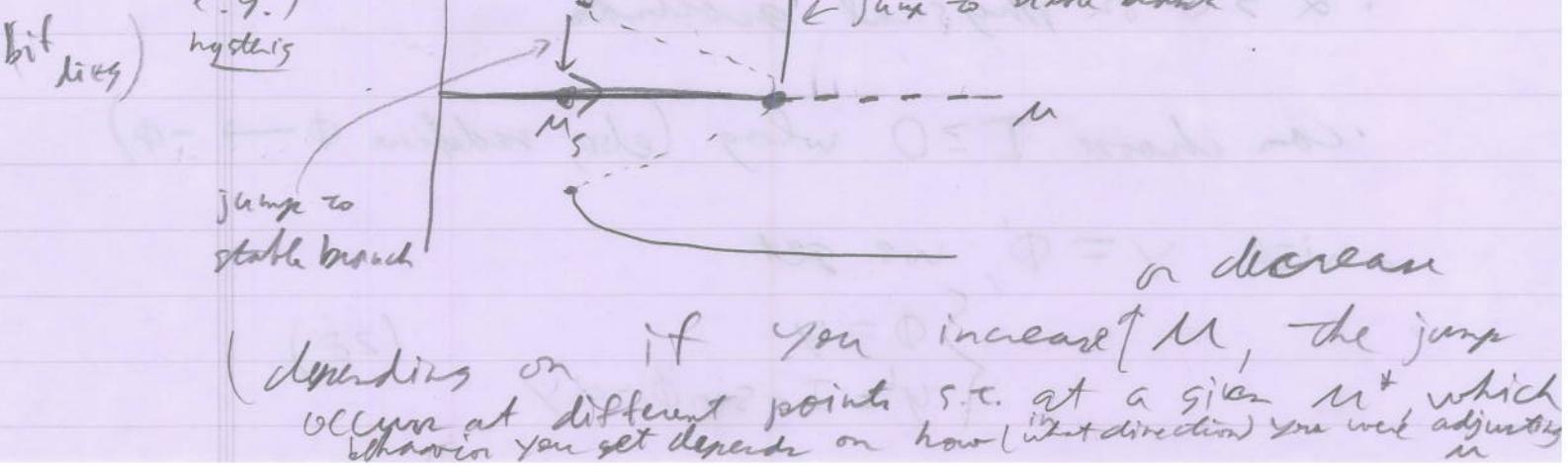
$$M > M_c$$

determined numerically

15a

MAP
1/14/10
B8b

- the key to this bif. is the behavior of the unstable manifold of the saddle; after it loops around, it either hits the origin (connecting to stable manifold & forming a homoclinic orbit; center) or veers off to one side or the other (left, right)
- e.g.) hysteresis in driven pendulum & Josephson junction
 - both homoclinic & infinite-period bif. arise
 - Same ~~equation~~ to use but multiple physical contexts ("the same equations have the same solutions")
 - for sufficiently weak damping, there are hysteresis effects because of coexistence of stable limit cycle & stable equilibrium



governing eqn (written in terms of Josephson junction):

$$\frac{\hbar C}{2e} \ddot{\phi} + \frac{\hbar}{2eR} \dot{\phi} + I_c \sin \phi = I_B, \quad (27)$$

where: \hbar = Planck's const. divided by 2π

e = charge of electron

I_B = constant bias current

C = junction's capacitance

R = junction's resistance

I_c = junction's critical current

$\phi(t)$ = phase difference across junction

- we nondimensionalize (1) using:

$$\tilde{\tau} = \left(\frac{2eI_c}{\hbar C} \right)^{1/2} t, \quad I = \frac{I_B}{I_c}, \quad \alpha = \left(\frac{\hbar}{2eI_c R^2 C} \right)^{1/2}$$

- (27) then becomes $\dot{\phi}'' + \alpha \dot{\phi}' + \sin \phi = I \quad (' = \frac{d}{d\tilde{\tau}})$

↳ (henceforth, I'll use τ for $\tilde{\tau}$)

some eqn as for a forced,
damped pendulum w/
const. forcing

- α = dimensionless damping

- I = dimensionless applied current

- $\alpha > 0$ on physical grounds

- can choose $I \geq 0$ wlog (else, redefine $\phi \rightarrow -\phi$)

- with $y = \phi'$, we get

$$\begin{cases} \dot{\phi}' = y \\ y' = I - \sin \phi - \alpha y \end{cases} \quad (28)$$

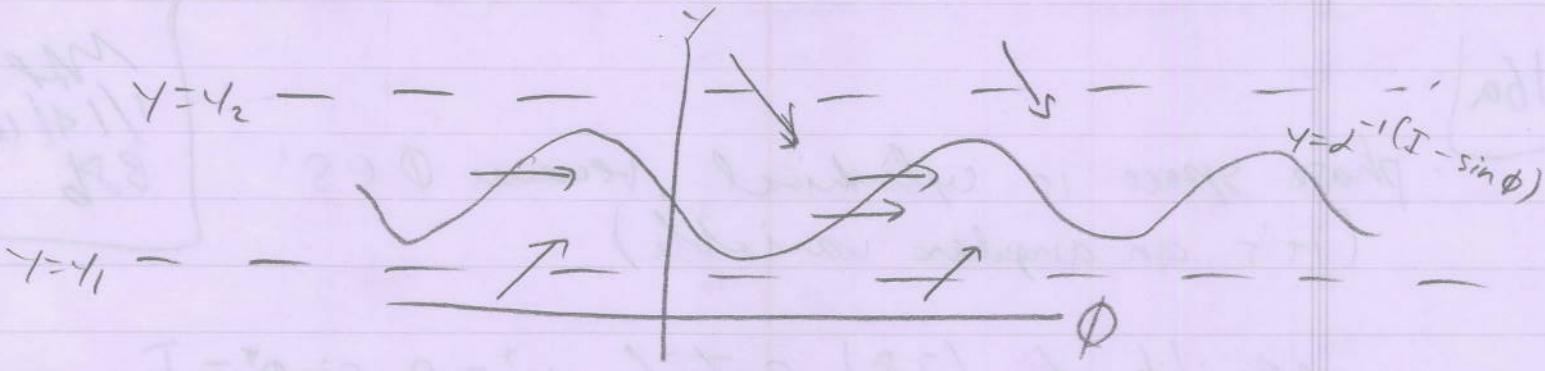
MAP
1/14/10
B86

- phase space is cylindrical because $\phi \in S^1$
(it's an angular variable)
- equilib of (28) satisfy $y^* = 0, \sin\phi^* = I$
 \Rightarrow 2 equilibria for $I < 1$ & none for $I > 1$
- $J = \begin{pmatrix} 0 & 1 \\ -\omega\phi^* - \alpha \end{pmatrix}$ can show that (when they exist) one equilib is a saddle & the other is stable

It's a node when $\lambda^2 - 4\sqrt{1-I^2} > 0$ (i.e., damping is strong enough or if $I \approx 1$)
It's a spiral when $\lambda^2 - 4\sqrt{1-I^2} < 0$

- at $I=1$, we have a saddle-node bif.
- claim: For $I > 1$, all trajectories are attracted to a unique stable limit cycle
- step 1: show that a periodic sol exists (this comes from a clever idea of Poincaré's)

Consider the nullcline $y = \lambda^{-1}(I - \sin\phi)$, where $y' = 0$. The flow is downward above the nullcline & upward below it



\Rightarrow all trajectories eventually enter the strip $y_1 \leq y \leq y_2$ & stay there forever

$(y_1 \text{ & } y_2 \text{ are any fixed numbers s.t. } 0 < y_1 < \frac{j-1}{2},$
 $y_2 > \frac{j+1}{2})$

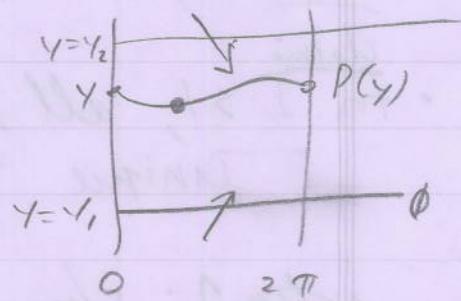
- inside the strip, flow is always to the right
 $(y' > 0 \Rightarrow \dot{\phi} > 0)$

- because $\phi=0$ & $\phi=2\pi$ are equivalent on the cylinder (recall $(y, \phi) \in \mathbb{R} \times S^1$, not simply $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$),

we need consider only the rectangular box

$$0 \leq \phi \leq 2\pi, y_1 \leq y \leq y_2$$

(technically, only need one of 0 & 2π , so say $0 \leq \phi < 2\pi$)



- suppose we start at height y at $\phi=0$ & follow it until we get to height $P(y)$ at $\phi=2\pi=0$
- the mapping $y \mapsto P(y)$ is called a Poincaré map (more details on P. map later!)

- it gives the height of the trajectory after one lap around the cylinder

- the Poincaré map is also called a "first-return map"

- we can't compute $P(y)$ explicitly (one rarely can), but if we can show that \exists a point y^* s.t. $P(y^*) = y^*$ (a fixed point), then the corresponding trajectory is a closed orbit

- to show that such a y^* exists, we consider what $P(y^*)$ looks like; consider a trajectory that starts at $y=y_1, \phi=0$; we claim that $P(y_1) > y_1$,

follows from fact that flow is strictly upward at first & trajectory can never return to line $y=y_1$, (as the flow is upward everywhere on that line)

- Same argument $\xrightarrow{\text{essential}} P(y_2) < y_2$

- now, $P(y)$ is continuous — this follows from a theorem that solutions of ODEs depend continuously on initial conditions if the "vector field" (aka, RHS of the dynamical system written down as a vector) is sufficiently smooth

- $P(y)$ is also monotonic (else, 2 trajectories would cross, which can't happen)

- So, $P(y)$ looks like this:

- intermediate value theorem
(alternatively: common sense!)

$$\Rightarrow P(y) = y \text{ somewhere,}$$

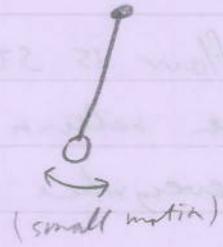
which gives the desired y^* and hence the closed orbit
(this proves existence)

- we actually almost have uniqueness as well, but we haven't excluded the possibility that $P(y) = y$ over an entire interval

Now we use the fact that there are two topologically distinct kinds of periodic orbits on the cylinder ($= \mathbb{R} \times S^1$):
librations & rotations



libration



(small motion)

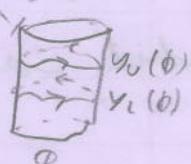


rotation



- but librations are impossible for $I > 1$, because any libration must encircle an equilib (can be shown using "index theory", but you should also be able to see this by thinking about it) & \exists equilib when $I > 1$

- suppose \exists 2 different rotations s.t. the phase portrait looks like this



- one rotation would need to lie strictly above the other one (because trajectories can't cross — RHS of vector field must specify uniquely when $\omega_2 = \omega_1$)
- $y_u(\phi)$ denotes upper rotation; $y_l(\phi)$ denotes lower rotation & $y_u(\phi) > y_l(\phi) \forall \phi$
- Now will show that the existence of 2 such orbits gives a contradiction:

we have the energy $E = \frac{1}{2}y^2 - \cos\phi$; after one circuit around any rotation $y(\phi)$, the change in energy ΔE must vanish

$$\Rightarrow 0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\phi} d\phi$$

$$\hookrightarrow \frac{dE}{d\phi} = y \frac{dy}{d\phi} + \sin\phi, \quad \frac{dy}{d\phi} = \frac{y'}{\phi'} = \frac{I - \sin\phi - dy}{y}$$

$$\Rightarrow \frac{dE}{d\phi} = I - dy \Rightarrow 0 = \int_0^{2\pi} (I - dy) d\phi \quad \forall \text{ rotation } y(\phi)$$

• equivalently, any rotation satisfies $\int_0^{2\pi} y(\phi) d\phi = \frac{2\pi I}{2}$ (29)

\hookrightarrow however, as $y_u(\phi) > y_l(\phi)$, we have $\int_0^{2\pi} y_u(\phi) d\phi > \int_0^{2\pi} y_l(\phi) d\phi$, so (29) can't hold for both rotations

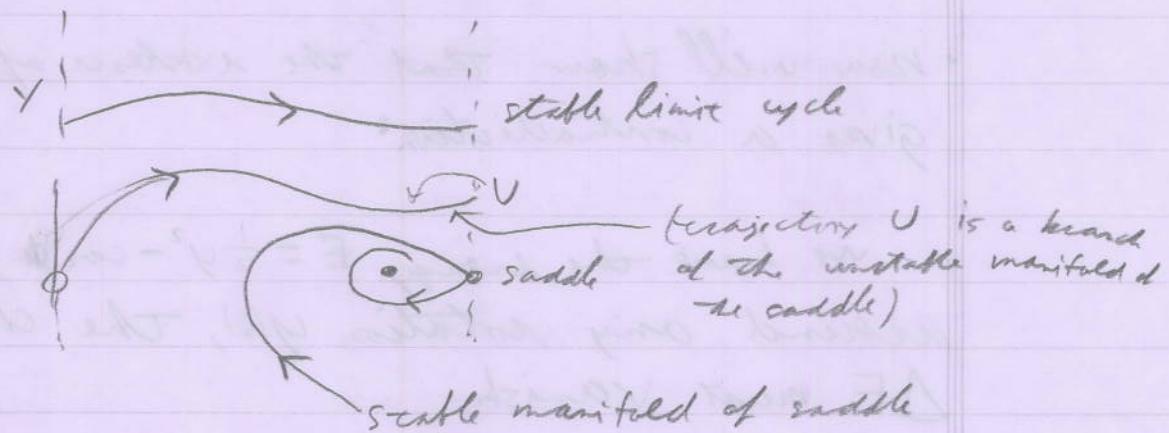
\therefore rotation for $I > 1$ is unique

• suppose we slowly decrease I starting from some value $I > I_c$

↳ think in terms of pendulum, which would have more & more trouble making it over the top; at some critical value $I_c < I$, the pendulum can no longer make it & it will damp to the rest state

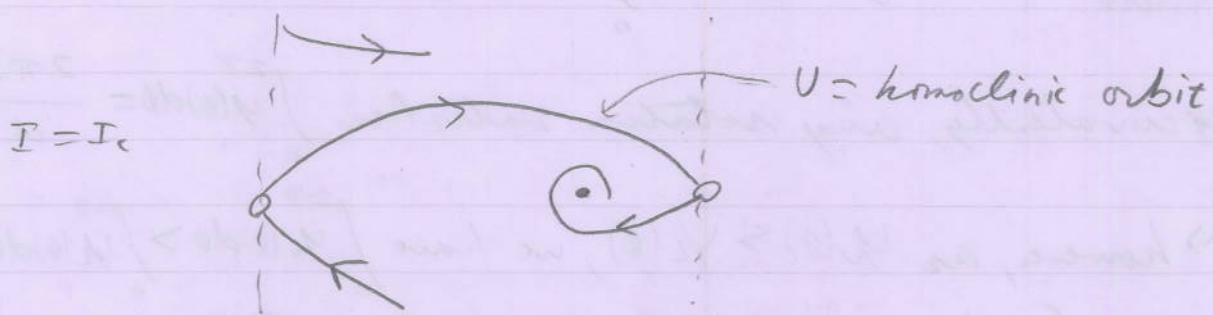
↳ we want to see what this looks like in phase space

• $I_c < I < 1$:



• trajectory $U \rightarrow$ stable limit cycle as $t \rightarrow \infty$

• as I decreases, stable limit cycle moves down & squeezes U closer to stable manifold of saddle; when $I = I_c$, limit cycle merges with U in a homoclinic bifurcation:

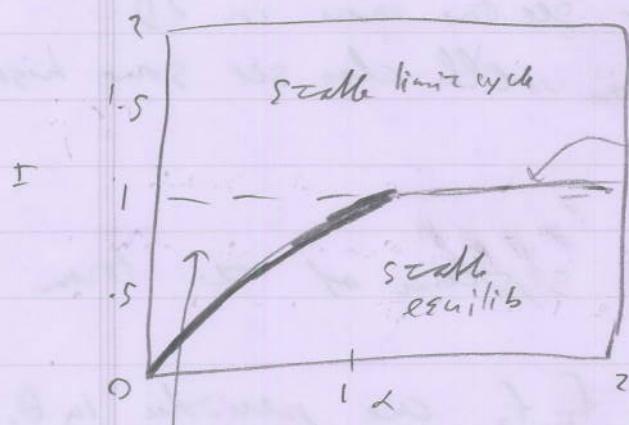


• when $I < I_c$, the saddle connection breaks and U spirals into a stable equilib.



This scenario is only for sufficiently small damping &

- we have the following overall bif. diag:



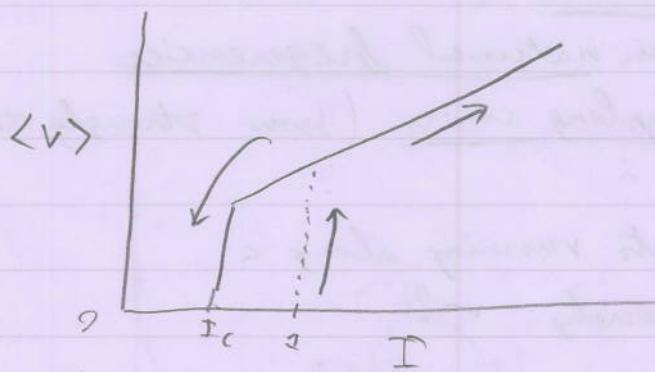
-- saddle-node bif
— intrinsic-period bif (transient)
— homoclinic bif.

"bistable" (= 2 stable equilibria).

- I presented a heuristic argument; Levi et al. (1978) has a rigorous proof; Buckenheimer & Holmes (reading list) derive an analytical approx for homoclinic bif curve for LCEI using "Melnikov's method"

hysteresis

- the above fig. explains why lightly damped Torsionbar junctions have hysteretic I-V curves (the similar result holds for pendula w/ the same parametrizing & damping)



Coupled Oscillators and Quasiperiodicity

line
+ circle

- thus far, we've seen phase space in 1D (\mathbb{R}^1, S^1) & 2D ($\mathbb{R}^2, \mathbb{R} \times S^1$); well now see one more in 2D (the torus, $S^1 \times S^1$) and then well also see some higher-dimensional situations this term
- phase space is $S^1 \times S^1$ for systems of the form

$$(30) \begin{cases} \dot{\theta}_1 = f_1(\theta_1, \theta_2) \\ \dot{\theta}_2 = f_2(\theta_1, \theta_2) \end{cases}, \text{ where } f_1, f_2 \text{ are periodic in } \theta_1 \text{ & } \theta_2$$

- called phase oscillators

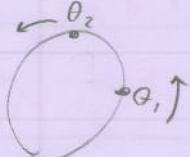
(which can occur, e.g., if we're on a limit cycle and want to use phase information to track where we are on the cycle)

$$\text{e.g.) } \begin{cases} \dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2) \end{cases} \quad (31) \quad \text{"coupled phase oscillators"}$$

where:

- θ_1, θ_2 are the phases of the oscillators (along a limit cycle)
- $\omega_1, \omega_2 > 0$ are their natural frequencies
- $K_1, K_2 \geq 0$ are coupling constants (how strongly they interact)

(image two points running along a circle simultaneously:



or, one point along a torus = a square with periodic b.c.'s



border
not same
are identified

- uncoupled case: $k_1 = k_2 = 0$

$$\Rightarrow \begin{cases} \dot{\theta}_1 = \omega_1 \\ \dot{\theta}_2 = \omega_2 \end{cases} \quad (\text{both oscillators run at their natural freq.})$$

- corresponding trajectories on the square are straight lines w/ const. slope $\frac{d\theta_2}{d\theta_1} = \frac{\omega_2}{\omega_1}$; there are 2 qualitatively different possibilities:

$$(1) \frac{\omega_1}{\omega_2} = \frac{p}{q} \text{ for } p, q \in \mathbb{Z} \text{ (integer)}$$

"rational"

\Rightarrow all trajectories are closed orbits

(A, completes p revolutions every time θ_2 completes q of them)

\rightarrow when plotted on the torus this give different types of "knots" for different p, q

(2)

$(2) \frac{\omega_1}{\omega_2} = \text{irrational}$; the flow is then said to be quasiperiodic (wavy trajectory winds endlessly on the torus, never intersecting itself & never closing)

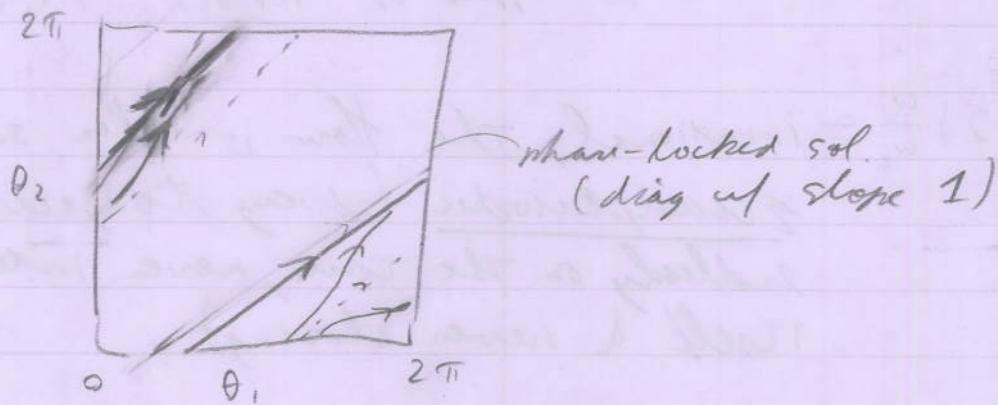
* this is a new type of long-term ("asymptotic") behavior

- coupled system w/ $k_1, k_2 > 0$

look at phase difference $\phi := \theta_1 - \theta_2$

$$\Rightarrow \dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2 - (k_1 + k_2) \sin \phi \quad (32)$$

- $\dot{\phi} = 0 \Rightarrow$
 • 2 equilibria if $|\omega_1 - \omega_2| < K_1 + K_2$
 but none if $|\omega_1 - \omega_2| > K_1 + K_2$
 (there's a saddle-node bif for $|\omega_1 - \omega_2| = K_1 + K_2$)
 ↳ (in (31), this means a saddle-node bif. of cycles)
- the equilibria (say we're in the case w/ 2) are defined implicitly by $\sin\phi^* = \frac{\omega_1 - \omega_2}{K_1 + K_2}$
 - all trajectories asymptotically approach the stable node, which corresponds to a phase-locked solution of (31)
 (a ↑ type of synchronization)
 - The sol. is periodic; both oscillators run at the const. freq. $\omega^* = \bar{\theta}_1 = \bar{\theta}_2 = \omega_2 + K_2 \sin\phi^* = \frac{K_1 \omega_1 + K_2 \omega_2}{K_1 + K_2}$
 (weighted avg. based on coupling strengths)



- if we're in the case w/o phase locking, then we have quasiperiodic or periodic flow (but w/ curves, not straight lines), depending on parameters

21a)

Poincaré Maps

(aka "Poincaré Sections")
"surface of section")

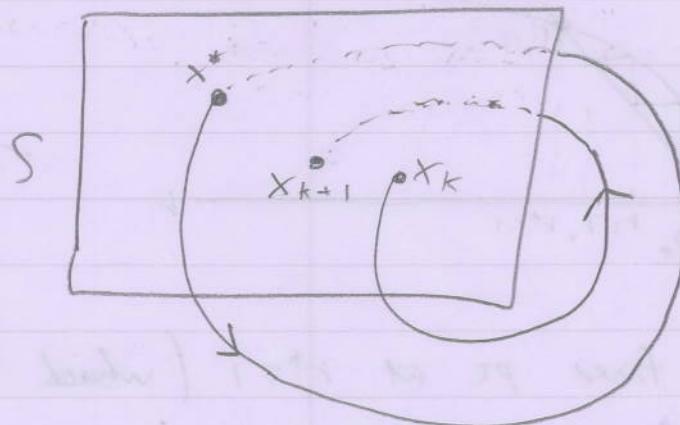
MAP
1/15/10
B8b

- used for studying flows by turning them into maps & can be especially helpful for making sense of complicated behavior (such as chaotic dynamics, which we'll see w/ 3D dynamical systems & forced 2D systems)

- $\dot{\vec{x}} = \vec{f}(\vec{x})$ (n -dim)



- let S be an $(n-1)$ -dim surface of section (aka, Poincaré section)



- S must be "transverse" to the flow (i.e., trajectories flow through S , not parallel to it)
- the Poincaré map $P: S \rightarrow S$ is obtained by following trajectories from one intersection w/ S to the next

- $\vec{x}_{k+1} = P(\vec{x}_k)$

\uparrow
fixed pt.

- $\vec{x}^* = P(\vec{x}^*)$

\uparrow
fixed pt.

note: almost always
impossible to find
explicit formula for P ;
this is ordinarily
done numerically

Can determine stability of closed orbit by looking at behavior of P near \vec{x}^*

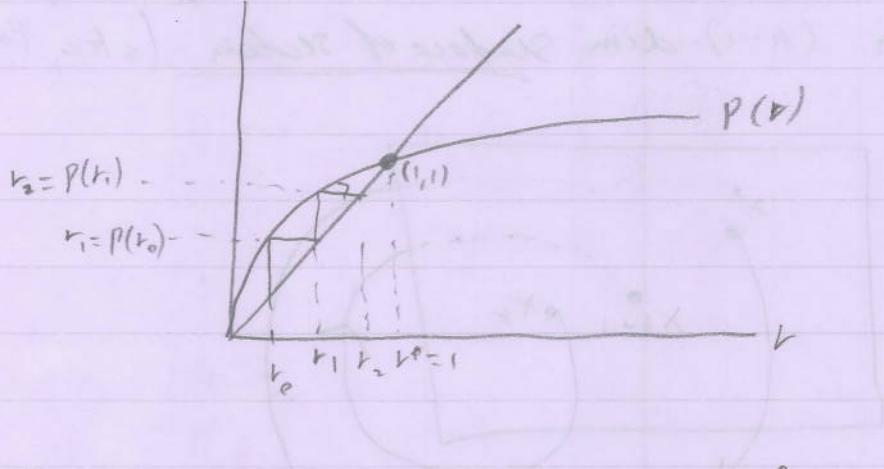
(corresponds to
a closed orbit of
original system)

$$\boxed{1b} \quad \text{e.g.) } \begin{cases} r = r(1-r^2) \\ \dot{\theta} = 1 \end{cases} \quad S = \text{positive } x\text{-axis}$$

- Let $r_0 \in S$ be initial pt.; $\dot{\theta} = 1 \Rightarrow$ first return to S occurs after time of flight $t = 2\pi$

$$\Rightarrow r_1 = P(r_0), \text{ where } r_1 \text{ satisfies } \int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi$$

$$\Rightarrow r_1 = [1 + e^{-4\pi}(r_0^{-2} - 1)]^{-1/2}, \text{ so } P(r) = [1 + e^{-4\pi}(r^{-2} - 1)]^{-1/2}$$



"cobweb" construction
(we use this to)
iterate P graphically

- P has a fixed pt at $r^* = 1$ (which is stable & unique)

Linear stability of periodic orbits

- $\dot{\vec{x}} = \vec{f}(\vec{x})$; given a closed orbit, how can we tell whether it's stable? (equivalently, is the fixed pt. \vec{x}^* stable?)

Note: $\vec{y} = P^2(\vec{y})$ means
(but $\vec{y} \neq P(\vec{y})$)
 \vec{y} is a fixed
pt. of P^2 &
hence a "period 2"
orbit of original
system; can we
progressively higher
periodic orbits (a so-called
"periodic orbit expansion") to
study chaotic dynamics (see Vicanovic book, to
which I link on the website)

- let \vec{v}_0 be a small perturbation s.t. $\vec{x}^* + \vec{v}_0 \in S$
- then $\vec{x}^* + \vec{v}_1 = P(\vec{x}^* + \vec{v}_0) = P(\vec{x}^*) + [DP(\vec{x}^*)]\vec{v}_0 + \underbrace{O(|\vec{v}_0|)}_{\text{small}}$
(after 1 return)

where $DP(\vec{x}^*)$ is an $(n-1) \times (n-1)$ [Jacobian] matrix called the linearized Poincaré map at \vec{x}^*

- because $\vec{x}^* = P(\vec{x}^*)$, we get $\vec{v}_1 = [DP(\vec{x}^*)]\vec{v}_0$.
- Stability criterion is expressed in terms of e-vals λ_j of $DP(\vec{x}^*)$: The closed orbit is linearly stable iff $|\lambda_j| < 1 \quad \forall j=1, \dots, n-1$
- Consider generic case in which there are no repeated e-vals $\Rightarrow \exists$ a basis of e-vects $\{\vec{e}_j\}$, so we can write $\vec{v}_0 = \sum_{j=1}^{n-1} v_j \vec{e}_j$ for some scalars v_j
 $\Rightarrow \vec{v}_1 = (DP(\vec{x}^*)) \sum_{j=1}^{n-1} v_j \vec{e}_j = \sum_{j=1}^{n-1} v_j \lambda_j \vec{e}_j$
- Iterating the linearized map k times gives
 $\vec{v}_k = \sum_{j=1}^{n-1} v_j (\lambda_j)^k \vec{e}_j$
 \therefore if all $|\lambda_j| < 1$, then $\|\vec{v}_k\| \rightarrow 0$ geometrically fast, which shows that \vec{x}^* is linearly stable
- conversely, if $|\lambda_j| > 1$ for some j , then perturbations along \vec{e}_j grow, so \vec{x}^* is unstable
- if the largest e-val satisfies $|\lambda_n| = 1$, then we need to work harder (possibly by doing nonlinear stability analysis); this occurs, e.g., at bifurcations of periodic orbits or in the stable case for Hamiltonian systems, for which symmetry requires e-vals to occur in pairs $\{\lambda, -\bar{\lambda}\}$ so only stable case we can satisfy $|\lambda| = 1$

22b]. The λ_i are called characteristic multipliers or Floquet multipliers of the periodic orbit

(note: there always exists an additional trivial multiplier $\lambda \equiv 1$ corresponding to time-translations along periodic orbit)

- in general, λ_i almost always need to be found numerically
overdamped

Ex.) N -dim system of coupled Josephson junctions:

$$\dot{\phi}_i = \Omega + a \sin \phi_i + \frac{1}{N} \sum_{j=1}^N \sin \phi_j, \quad i = 1, \dots, N \quad (33)$$

in parallel resistive load (Tran et al, 1991)

- in-phase sol. is given by $\phi_1(t) = \phi_2(t) = \dots = \phi_N(t) \equiv \phi^*(t)$

for such solutions, (33) reduces to

$$(34) \quad \frac{d\phi^*}{dt} = \Omega + (a+1) \sin \phi^*, \text{ which has a periodic solution on the circle iff } |\Omega| > |a+1|$$

- to find the stability of the in-phase sol, we do "linear stability analysis" $\phi_i(t) = \phi^*(t) + \eta_i(t)$, where $\eta_i(t)$ are infinitesimal perturbations

↓
Substitute into (33) & drop quadratic terms in η_i to

$$\dot{\eta}_i = [a \cos \phi^*(t)] \eta_i + [\cos \phi^*(t)] \frac{1}{N} \sum_{j=1}^N \eta_j$$

- we don't have $\phi^*(t)$ explicitly, but that's ok because we have some tricks:

• linear system decouples if we change vars to
(first) $M := \sum_{j=1}^N \eta_j, \quad \xi_i := \eta_{i+1} - \eta_i \quad (i=1, \dots, N-1)$

23a

MAP
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BD

$$\Rightarrow \dot{\xi}_i = [a \omega_2 \phi^*(t)] \xi_i$$

• separation of vars gives $\frac{d\xi_i}{\xi_i} = [a \cos \phi^*(t)] dt = \frac{[a \cos \phi^*] d\phi^*}{R + (a+1) \sin \phi^*}$

where we used (34) to eliminate dt (that's the 2nd trick)

• now compute the change in the perturbations after one circuit around the closed orbit ϕ^* :

$$\oint \frac{d\xi_i}{\xi_i} = \int_0^{2\pi} \frac{[a \cos \phi^*] d\phi^*}{R + (a+1) \sin \phi^*} \quad (\text{T = period})$$

$$\Rightarrow \ln \left(\frac{\xi_i(T)}{\xi_i(0)} \right) = \frac{a}{a+1} \ln \left[R + (a+1) \sin \phi^* \right] \Big|_0^{2\pi} = 0$$

$\Rightarrow \xi_i(T) = \xi_i(0)$; we can similarly show $u(T) = u(0)$ so $\eta_i(T) = \eta_i(0) \forall i$ & all perturbations are unchanged after one cycle! \Rightarrow all $\lambda_i = 1$

\therefore in-phase mode is (linearly) neutrally stable
(need all $|\lambda_i| < 1$ for ^{linear} asymptotic stability)

- however, one needs to consider nonlinear terms to see more of what's going on

e.g.) N phase oscillators on a line with nearest-neighbor coupling

$$(35) \quad \begin{cases} \ddot{\theta}_1 = \omega_1 + \alpha \sin(\theta_2 - \theta_1) \\ \ddot{\theta}_i = \omega_i + \alpha [\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], i=2, \dots, N-1 \\ \ddot{\theta}_N = \omega_N + \alpha \sin(\theta_{N-1} - \theta_N) \end{cases}$$

- all coupling const. λ are equal but the natural freq ω_i can be different

- let $\phi_i = \theta_i - \theta_{i+1}$, $i=1, \dots, N-1$

\Rightarrow (35) can be written in matrix form:

$$\frac{d\vec{\phi}}{dt} = \vec{\omega} + A\vec{s}, \quad (36)$$

where: $\vec{\phi} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_{N-1} \end{bmatrix}$, $\vec{\omega} = \begin{bmatrix} \omega_1 - \omega_2 \\ \vdots \\ \omega_{N-1} - \omega_N \end{bmatrix}$, $\vec{s} = \begin{bmatrix} \sin \phi_1 \\ \vdots \\ \sin \phi_{N-1} \end{bmatrix}$,

$$A = \lambda \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 & -2 \\ 0 & & & & \ddots & \ddots \end{bmatrix}$$

is a tri-diagonal matrix

- equilibria are phase-locked orbits ($\dot{\phi}_i = 0 \forall i$)

- $\dot{\vec{\phi}} = 0 \Rightarrow \vec{s} = \underbrace{-A^{-1}\vec{\omega}}_{\text{for this to have a real sol., need each component of } A^{-1}\vec{\omega} \text{ to be } \leq 1} \quad (37)$

for this to have a real sol., need each component of $A^{-1}\vec{\omega}$ to be ≤ 1 (as all components of \vec{s} are sines)

- A can be inverted in closed form: A^{-1} is a symmetric matrix ($A_{ij} = A_{ji}$) w/ elements $(A^{-1})_{ij} = \frac{j(N-1)}{-N\lambda}$ ($i \neq j$)
 (get elements $i < j$ from sym)

- suppose that $\omega_1 = \omega$, $\omega_2 = \omega - \Delta$, $\omega_3 = \omega - 2\Delta$, etc

$$\Rightarrow \vec{\omega} = \begin{bmatrix} \omega_1 - \omega_2 \\ \vdots \\ \omega_{N-1} - \omega_N \end{bmatrix} = \Delta \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

24a

 MAP
 1/16/10
 B8b

- Δ = uncoupled freq. difference between 2 adjacent oscillators

$$\Rightarrow \text{at equilib, we have } \sin \phi_i = \frac{\Delta' i(N-i)}{2d} \quad i=1, \dots, N-1$$

$$\text{suppose we take } N=6; \text{ then } \Delta' = -\frac{1}{6d} \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$(37) \Rightarrow \vec{s} = \begin{bmatrix} \sin \phi_1 \\ \vdots \\ \sin \phi_5 \end{bmatrix} = \frac{\Delta}{2d} \begin{bmatrix} 5 \\ 8 \\ 9 \\ 8 \\ 5 \end{bmatrix}$$

requiring each sine term to have magnitude ≤ 1 gives the condition for phase locking: $\left| \frac{\Delta}{2} \right| \leq \frac{8}{N^2}$

$$= \frac{8}{36} = \frac{2}{9}$$

- again, the key for phase locking is the ratio of frequency difference to coupling strength

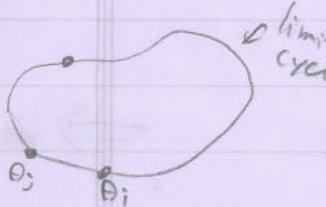
Synchronization

- Sync = an adjustment of rhythms of oscillating objects due to their (possibly extremely weak) interaction
 - e.g.) phase-locking (which we've seen) is the simplest type (give some details in lecture)
 - e.g.) Huygen's clocks, 'Crowd Sync on the Millennium Bridge'

- 246] (Kuramoto showed using "averaging" that)
 • any system of weakly coupled, nearly identical limit-cycle oscillators has long term ($t \rightarrow \infty$) dynamics given by phase equations of the following form:

$$(38) \dot{\theta}_i = \omega_i + \sum_{j=1}^N \Gamma_{ij}(\theta_j - \theta_i), \quad i=1, \dots, N$$

function only of phase difference between oscillators



- the functions Γ_{ij} can be calculated as integrals involving certain terms from original limit cycle model (which could be, e.g., a bunch of coupled - aka, interacting - van der Pol oscillator)
- describes dynamics after reaching limit cycle (so, the phase θ_i indicates where the oscillator is along its path)
- Kuramoto model: simplest case of equally-weighted, all-to-all, purely sinusoidal coupling:

$$(39) \Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} \sin(\theta_j - \theta_i),$$

where:

- $K \geq 0$ is coupling strength
- factor $\frac{1}{N}$ ensures good behavior as $N \rightarrow \infty$

- Note: all-to-all means every oscillator is affected by every oscillator; can be generalized by considering graphs describing which oscillators are affected by which (we've already seen nearest-neighbor coupling, but one can put something arbitrarily complicated here & get some really cool stuff!)

("networks", "synchronization on networks", "dynamics on networks", etc)

25a)

 MAP
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 B8b

- frequencies ω_i are distributed according to some probability density $g(\omega)$
- original case: $g(\omega) = \text{unimodal \& symmetric about mean frequency } \bar{\omega}$ (i.e., $g(\bar{\omega} + \omega) = g(\bar{\omega} - \omega) \forall \omega$; e.g., a Gaussian dist.)
- rotational sym. $\Rightarrow \bar{\omega} = 0$ wlog
(simply redefine $\theta_i \mapsto \theta_i + \bar{\omega}t \forall i$, so we're in a frame rotating w/ freq. $\bar{\omega}$)
- $\therefore \dot{\theta}_i = \omega_i + \frac{k}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i=1, \dots, N, \quad (40)$
- $g(\omega) = g(-\omega) \quad (\text{i.e., sym \& mean}=0)$
- Unimodal $\Rightarrow g(\omega)$ is nowhere increasing on $[0, \infty)$, so that $g(\omega) \geq g(v)$ whenever $\omega \leq v$

- order parameter (the idea of order params)
 - ↳ a way of measuring how much 'order' a system has
 - ↳ such a number might measure, e.g., how much alignment there is in a chain of magnets, etc.
- q/lx order param:

$$\underbrace{re^{i\psi}}_{\substack{\downarrow \\ \text{a macroscopic quantity that}}} = \frac{1}{N} \sum_{i=1}^N e^{i\theta_i} \quad (41)$$



- ↳ can be interpreted as the collective rhythm produced by the population of oscillators
- $r(t)$ = "coherence", $\psi(t)$ = average phase

- e.g., if all oscillators moving in single dump $\Rightarrow r \approx 1$, so population acts like one giant oscillator
- if oscillators scattered throughout circle, then $r \approx 0$, as the individual oscillations add incoherently & no macroscopic rhythm is produced
- so the order param. provides a way to measure the degree of synchrony

• multiply (41) by $e^{-i\theta_i}$ to get $re^{i(\psi - \theta_i)} = \frac{1}{N} \sum_{j=1}^N e^{i(\theta_j - \theta_i)}$ (42)

• equating imaginary parts $\Rightarrow r \sin(\psi - \theta_i) = \frac{1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)$

\hookrightarrow so (40) becomes $\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i), i=1, \dots, N$ (43)

oscillators interact only through "mean-field" quantities r, ψ

(i.e., each experiences the other oscillators only through their averaged properties)

- θ_i is pulled towards mean phase ψ
- effective strength of coupling is proportional to coherence r

• sync mechanism in this model: positive feedback loop between coupling strength and coherence

- try simulating the Kuramoto model at home!
(lots of cool behavior can happen...)

26a]

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B8b

$N \rightarrow \infty$ limit

- analysis for Kuramoto model is hard, but one can gain insights with $N \rightarrow \infty$ limit

- seek sol. w/ $r(t) = \text{const.}$ & $\Psi(t)$ rotates uniformly at freq Ω_r (in rotating frame, $\dot{\Psi} = 0$ vlog)

$$\Rightarrow \dot{\theta}_i = \omega_i - Kr \sin \theta_i, i=1, \dots, N \quad (44)$$

T
resonate for this type of sol, so all oscillators decouple, so we can study this easily

- strategy: solve (44) for motion of all oscillators (which depend on r as a param) \Rightarrow values of r, Ψ must be consistent w/ those assumed originally, which gives a key self-consistency condition

- 2 cases:

- ① $|\omega_i| \leq Kr \Rightarrow$ oscillators approach a stable equilib defined implicitly by $\omega_i = Kr \sin \theta_i$
(where $|\theta_i| \leq \frac{\pi}{2}$)

- these oscillators are 'locked' because they are phase-locked at freq. Ω_r in original frame

- ② $|\omega_i| > Kr \Rightarrow$ these oscillators are "drifting" — they run around circle in nonuniform manner
- the faster oscillators keep lapping the locked oscillators & the slower ones keep being lapped by them
 \leftarrow
- accelerating in some places & slowing down at others

26b] - locked oscillators are in center of $g(\omega)$ & drifting ones are in the tails

- drifting oscillator from a stationary distribution on the circle \Rightarrow centroid $\stackrel{\text{hence it's}}{\rightarrow}$ stays fixed even though oscillators continue to move

$$\rho(\theta, \omega) d\theta = \text{fraction of oscillators w/ } \omega \in [\theta, \theta + d\theta]$$

- stationarity $\Rightarrow \rho(\theta, \omega)$ is inversely proportional to speed at θ
 $\begin{cases} \text{oscillators} \\ \text{pile up in slow places & spread out in fast places} \end{cases}$

$$\hookrightarrow \rho(\theta, \omega) = \frac{C}{|w - Kr \sin \theta|}, \quad \text{where}$$

$$\int_{-\pi}^{\pi} \rho(\theta, \omega) d\theta = 1 \Rightarrow C = \frac{1}{2\pi} \sqrt{w^2 - (Kr)^2}$$

- now we invoke self-consistency: values of r, γ must be consistent w/ those defined by (41)

$$\Rightarrow \langle e^{i\theta} \rangle = \langle e^{i\theta} \rangle_{\text{lock}} + \langle e^{i\theta} \rangle_{\text{drift}}, \quad \text{where } \langle \rangle \text{ denotes population averages}$$

$$\cdot \underbrace{\gamma = 0}_{\text{by assumption, so } \langle e^{i\theta} \rangle = re^{i\gamma} = r}$$

$$\Rightarrow r = \langle e^{i\theta} \rangle_{\text{lock}} + \langle e^{i\theta} \rangle_{\text{drift}}$$

$$\langle \rangle = \frac{1}{M} \sum^M (\) \quad \left(\rightarrow \text{an integral in limit as } M \rightarrow \infty \right)$$

- lock: in the locked state, $\sin \theta^* = \frac{\omega}{Kr} \quad \forall |w| \leq Kr$; as $N \rightarrow \infty$, the distribution of locked phases is symmetric about $\theta = 0$ because $g(\omega) = g(-\omega)$

$$\Rightarrow \langle \sin \theta \rangle_{\text{lock}} = 0, \quad \text{so} \quad \langle e^{i\theta} \rangle_{\text{lock}} = \langle \cos \theta \rangle_{\text{lock}} = \int_{-Kr}^{Kr} \cos[\theta(\omega)] g(\omega) d\omega$$

27a)

 Msp
 1/16/10
 B8b

- changing var from ω to θ gives

$$\langle e^{i\theta} \rangle_{\text{lock}} = \int_{-\pi/2}^{\pi/2} (\cos \theta) g(Kr \sin \theta) (Kr \cos \theta) d\theta = Kr \int_{-\pi/2}^{\pi/2} (\cos^2 \theta) g(Kr \sin \theta) d\theta$$

• drift: they contribute $\langle e^{i\theta} \rangle_{\text{drift}} = \int_{-\pi}^{\pi} \int e^{i\theta} \rho(\theta, \omega) g(\omega) d\omega d\theta$

$$= 0 \text{ because } g(\omega) = g(-\omega)$$

$\& e^{i(\theta + \pi)} = -e^{i\theta}$

- the self-consistency condition is

$$r = Kr \int_{-\pi/2}^{\pi/2} (\cos^2 \theta) g(Kr \sin \theta) d\theta \quad (46)$$

- $r=0$ is a sol to (46) $\forall K$
 \Rightarrow completely incoherent state w/ $\rho = \frac{1}{2\pi} \delta(\omega)$

- a second branch of solutions ("partially synchronized states") satisfies

$$1 = K \int_{-\pi/2}^{\pi/2} (\cos^2 \theta) g(Kr \sin \theta) d\theta \quad (47)$$

- this branch bifurcates continuously from $r=0$ at a value $K=K_c$ obtained by letting $r \rightarrow 0^+$ in (47)

$$\Rightarrow K_c = \frac{2}{\pi g(0)} \quad \left(\begin{array}{l} \text{so we start seeing sync} \\ \text{at this coupling strength;} \\ \text{check numerically!} \end{array} \right)$$

- expanding integrand in (47) in powers of r reveals that bif is supercritical for $g''(0) < 0$ & subcritical for $g''(0) > 0$

27b

- near onset, the amplitude of the bifurcating branch obey

$$r \approx \sqrt{\frac{16}{\pi K_c^3}} \sqrt{\frac{u}{-g''(0)}}, \text{ where } u = \frac{K - K_c}{K_c}$$

- special case: "Lorentzian" (aka "Cauchy") density

$$g(w) = \frac{Y}{\pi(Y^2 + w^2)}, \quad (47) \text{ can be integrated}$$

exactly to yield $r = \sqrt{1 - \frac{K_c}{K}} \quad \forall K \geq K_c$

- Puzzles (think about them!):

1) finite- N fluctuations? (analysis above work $N \rightarrow \infty$)

2) stability of synchronized state?

(do some computations to take a look)

Normal Forms

- we often linearize systems to look at local behavior, but sometimes the behavior of the linear system is entirely different from the nonlinear one & no info is obtained from linear analysis (i.e., there is crucial info in the nonlinear terms)

- normal forms theory is a way to try to extract this information (e.g., each bifurcation has a normal form corresponding to the platonic ideal version of that bifurcation)

28a)

- symmetric \Rightarrow an equilib at $(0,0)$

MAP
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B8b

consider

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2) \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2) \end{cases} \quad (48)$$

- f_1, f_2 are nonlinear, smooth & their Taylor expansion about the origin has no linear terms

- linear stability analysis \Rightarrow look at $\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned}$

Δ find eigenvalues λ_1, λ_2 of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
(well assume A is diagonalizable)

- if $\text{Re } \lambda_1, \text{Re } \lambda_2 \neq 0$, we can get necessary info from linear stability analysis, but we can't if $\text{Re } \lambda_i = 0$
- then all of the stability info is contained in the nonlinear terms

- e.g.)
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \alpha x_1^2 x_2 \end{cases}$$
 ← stability of $(0,0)$
depends on the sign of α

- step 1 in normal form analysis of (48) is to find a linear change of vars. set

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 + g_1(y_1, y_2) \\ \dot{y}_2 = \lambda_2 y_2 + g_2(y_1, y_2) \end{cases} \quad (49)$$

- step 2: seek a near-identity transformation in the form $\begin{cases} y_1 = z_1 + P_1(z_1, z_2) \\ y_2 = z_2 + P_2(z_1, z_2) \end{cases}$ (50)

where P_1, P_2 are power series

- the choice of (50) is meant to simplify (49)

- expand g_1, g_2 in power series & choose the coeffs. of P_1, P_2

s.t. the system $\begin{cases} \dot{z}_1 = \lambda_1 z_1 + h_1(z_1, z_2) \\ \dot{z}_2 = \lambda_2 z_2 + h_2(z_1, z_2) \end{cases}$ (51)

is simpler than (49)

- optimally, $h_1 = h_2 = 0$, which corresponds to an exact linearization of the original system but in general some nonlinear terms will remain

- the ability to linearize the system is connected to the e-vale λ_1, λ_2

- if either $(n_1 - 1)\lambda_1 + n_2\lambda_2 = 0$ or $n_1\lambda_1 + (n_2 - 1)\lambda_2 = 0$ for some positive integers n_1, n_2 , then the eigenvalues are said to be resonant (or in resonance) & one of the functions h_1, h_2 contains resonant terms of the form $z_1^{n_1} z_2^{n_2}$ (that can't be eliminated by the near-identity transformation)

(if no resonance, we can do exact linearization)

- if λ_1, λ_2 are pure imaginary, $z_2 = \bar{z}_1 \equiv \bar{z}$ & there are ∞ many resonant conditions
↳ the stability of $(0,0)$ is then determined by the first nonzero coeff of h_1 .