

29a

 MAP  
 1/17/10  
 BS3

$$\text{e.g.) } \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_1 = -x_1 + \alpha x_1^2 x_2 \end{cases} \quad (52)$$

 chose to  
 diagonalize  
 linear part

- linear change of vars.  $\begin{cases} x_1 = y_1 + y_2 \\ x_2 = i(y_1 - y_2) \end{cases}$  gives

$$\begin{cases} \dot{y}_1 = iy_1 + \frac{\alpha}{2}(y_1^2 y_2 - y_1 y_2^2 - y_1^3 + y_2^3) \\ \dot{y}_2 = -iy_2 - \frac{\alpha}{2}(y_1^3 - y_1^2 y_2 + y_1 y_2^2 + y_2^3) \end{cases} \quad (53)$$

- The normal form transformation to 3rd order is

$$\begin{cases} y_1 = z_1 + \frac{i\alpha}{8}(2z_1^3 + 2z_1 z_2^2 - z_2^3) + \text{h.o.t.} \\ y_2 = z_2 + \frac{i\alpha}{8}(z_1^3 - 2z_1^2 z_2 - 2z_2^3) + \text{h.o.t.} \end{cases} \quad (54)$$

(h.o.t. = higher order terms, which here mean 4th power & higher)

$\therefore$  normal form is

$$\begin{cases} \dot{z}_1 = iz_1 + \frac{\alpha}{2}z_1 z_2^2 + \text{h.o.t.} \\ \dot{z}_2 = -iz_2 + \frac{\alpha}{2}z_2 z_1^2 + \text{h.o.t.} \end{cases} \quad (55) \quad \checkmark \text{ cplx conj.}$$

- (55) gives  $\ddot{\rho} = \frac{\alpha}{2}\rho^3 + \text{h.o.t.}$  (where  $\rho^2 = z_1 z_2$ )

$\Rightarrow (0,0)$  is stable for  $\alpha < 0$  & unstable for  $\alpha > 0$

(we have (neutral) stability for  $\alpha = 0$  because that's just a linear system)

- if coeff. of normal form all vanish, then equilib. is surrounded by a family of periodic orbits (i.e., it's a nonlinear center)

ib]. We can do the same procedure for  $n$ -dim dynamical systems

- normal forms are useful for studying bifurcations as it's a way of seeing their platonic ideal form & of 'zooming in' to see what's going on there
- there are several alternatives to normal forms that basically accomplish the same thing (e.g., "amplitude eqn")
- a brief note on bifurcation

we have seen a few types of bifurcations (and some global ones), but they can be arbitrarily complicated

- The codimension of a bifurcation is the smallest dim of parameter space that contains the bifurcation in a persistent way
- The last bifurcations we have seen have codimension 1, which can occur when one has a single zero  $(J = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix})$  or a single pure imaginary pair  $\begin{array}{l} \uparrow \\ \text{some matrix} \end{array} (J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix})$

• For codimension 2 and above, much more complicated things can occur; codim 2 occurs for:

- double zero, non-diagonalizable;  $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- simple zero + pure imaginary pair;  $J = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- two pure imaginary pairs

# Maps

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B8b

- Maps: discrete-time dynamical systems  
(we've already seen Poincaré maps)

• L.S.,  $x_{n+1} = f(x_n)$  <sup>(56)</sup> is a 1D map, which can already exhibit very complicated behavior

$\rightarrow$  <sup>(52)</sup>  $\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$  is a 2D map

(note: I'll use "map" to refer both to (56) & to  $f$  by itself.)

- The sequence of points  $x_0, x_1, x_2, \dots$  starting from  $x_0$  is the orbit of  $f$  starting at  $x_0$ .

- Fixed Points & Cobwebs

- we'll start w/ 1D maps of the form (56), where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is smooth

- $x^*$  s.t.  $f(x^*) = x^*$  is a fixed point

- to determine its stability, we look at a nearby orbit given by  $x_n = x^* + \gamma_n$  & ask if it is attracted to or repelled from  $x^*$

- Substitution  $\Rightarrow x^* + \gamma_{n+1} = x_{n+1} = f(x^* + \gamma_n) = f(x^*) + f'(x^*)\gamma_n + O(\gamma_n^2)$

$$\Rightarrow \gamma_{n+1} = f'(x^*)\gamma_n + O(\gamma_n^2)$$

$\uparrow$   
because  $f(x^*) = x^*$

- neglecting  $O(\eta_n^2)$  term, we get a linearized map  $\eta_{n+1} = f'(x^*)\eta_n$   
 with eigenvalue (or multiplier)  $\lambda = f'(x^*)$

↪ sol. of linearized map can be found explicitly:

$$\cdot \eta_1 = \lambda \eta_0, \eta_2 = \lambda \eta_1 = \lambda^2 \eta_0, \dots, \eta_n = \lambda^n \eta_0.$$

• if  $|\lambda| = |f'(x^*)| < 1$ , then  $\eta_n \rightarrow 0$  as  $n \rightarrow \infty$  &  
 $x^*$  is linearly stable for original map

• if  $|\lambda| > 1$ , the fixed pt. is unstable for original map

• in marginal case of  $|\lambda|=1$ , then we need the  $O(\eta_n^2)$  terms to determine stability

• note: all of these results have parallels for ODEs that you've seen before (in Part A)!!

$$\text{e.g., } x_{n+1} = x_n^2$$

• fixed points satisfy  $x^* = (x^*)^2 \Rightarrow x^* = 0, 1$

$$\cdot \lambda = f'(x^*) = 2x^* = \begin{cases} 0, & x^* = 0 \\ 2, & x^* = 1 \end{cases}$$

stable  
unstable

this case  
of  $\lambda=0$  is  
sometimes  
called  
superstable

Because  
perturbations  
decay like  
fast  
 $(y_n - y_0)^{2^n}$ ,  
ordinary  
stable pt.  
has decay  
 $y_n - \lambda^n y_0$

• recall that we can iterate maps using whatever; can we use this technique to show, e.g., that  $x_{n+1} = \sin x_n$  has a globally stable fixed pt. at  $x=0$  (exercise)

3/a

## Logistic Map

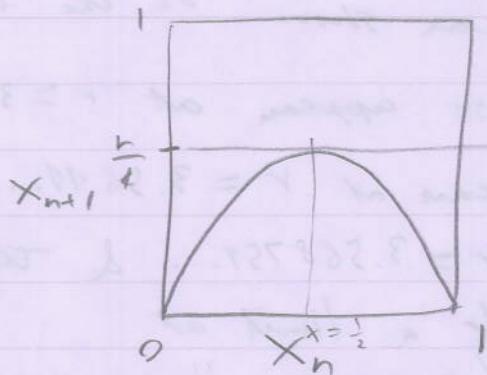
MAP  
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BSb

- 1976 <sup>review</sup> article by Robert May (from here)
  - illustrated that even seemingly very simple nonlinear maps can have very complicated dynamics
  - he illustrated his pt. with logistic map

$$x_{n+1} = r x_n (1 - x_n) \quad (58), \text{ a discrete-time analog of logistic eqn for population growth}$$

$$= f(x_n).$$

- in this app,  $\begin{cases} x_n \geq 0 \text{ is dimensionless measure of pop in } n^{\text{th}} \text{ generation} \\ r \geq 0 \text{ is its intrinsic growth rate} \end{cases}$

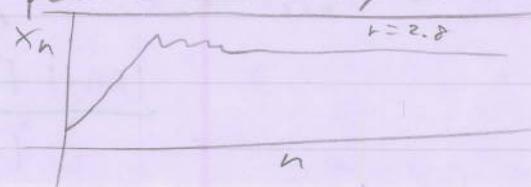


- will let  $r \in [0, 1]$  s.t.  
 $f: [0, 1] \rightarrow [0, 1]$   
 (most interesting behavior occurs here — try this on the computer!)

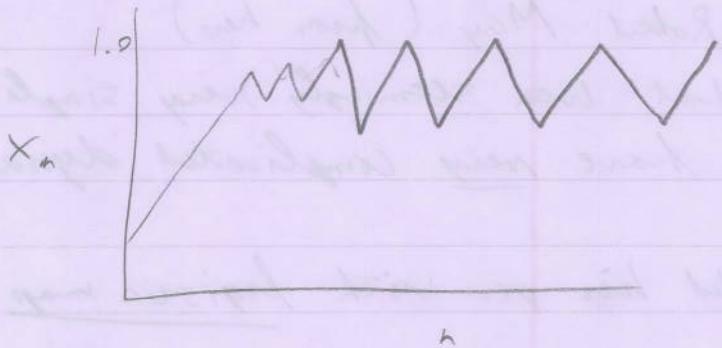
### period-doubling

- fix  $r$ , some initial pop  $x_0$ , & iterate (58)
- $r < 1 \Rightarrow x_n \rightarrow 0$  as  $n \rightarrow \infty$  (show by overwrapping)

- $1 < r < 3 \Rightarrow x_n$  grows steadily & reaches non-zero steady state



- larger  $r$  (e.g.,  $r = 3.3$ ), the population increases until it reaches steady state period-2 oscillation (period 2 = fixed pt. of map  $f^2$ ): ("period 2 cycle")



- for large  $r$  (e.g.,  $r = 3.5$ ), we build up to period 4 cycles (I can't draw this anymore, so try this at home — also look up "period doubling route to chaos".)
- we get further period doublings as we increase  $r$  further, & you can show on the computer that period 8 first appears at  $r = 3.54409\dots$ , period 16 first appears at  $r = 3.5641\dots$ , period 32 first appears at  $r = 3.568759\dots$  & that there appear to be a limit at  $r_\infty = 3.569946\dots$  (Successive "period-doubling bifurcations" occur increasingly fast)
- The convergence is essentially geometric; in the limit of large  $N$ , the distance between successive transitions shrinks by a factor of

$$\delta = \lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\dots$$

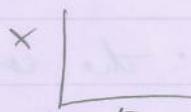
$\uparrow$   
 (first) Feigenbaum constant  
 (more on this later)

- $r > r_{\text{co}}$ : it's complicated
  - for many values of  $r$ , the sequence  $\{x_n\}$  never settle to a fixed pt. or periodic orbit
    - instead,  $r \rightarrow \infty$  behavior is aperiodic (not periodic, this is a discrete-time version of chaos)
    - look in Matlab what the colored diag looks like)
  - need to see behavior of all values of  $r$  at once; do this with an orbit diag (a type of bit. diag).

pseudocode:

for

each value of  $r$



- pick  $x_0$  at random
- iterate many times (say, 300) to allow system to settle to long-time behavior
- plot more points after  $x_{300}$  (say,  $x_{301}, \dots, x_{600}$ ) above  $x_{300}$

end

$\Rightarrow$  a very famous diag w/ both periodic & chaotic windows (e.g. large window near  $r \approx 3.83$  contains a stable period 3-cycle)

- Tim Gowers
- Li-Yorke (1975), "Period Three Implies Chaos" (invention of word chaos for math...)

- now let's do some analytical stuff

- $X_{n+1} = r X_n (1 - X_n)$ ,  $0 \leq X_n \leq 1$ ,  $0 \leq r \leq 4$

- find all fixed pts. & determine their stability

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$$x^* = f(x^*) = rx^*(1-x^*) \Rightarrow x^* = 0 \text{ or } 1 = r(1-x^*)$$

$$\Rightarrow x^* = 1 - \frac{1}{r} \text{ (allowable only if } r \geq 1)$$

- eigenvalue  $\lambda = f'(x^*) = r - 2rx^*$

- $f'(0) = r \Rightarrow \lambda^* = 0$  stable for  $r < 1$  & unstable for  $r > 1$

- $f'(1 - \frac{1}{r}) = r - 2r(1 - \frac{1}{r}) = 2 - r \Rightarrow \lambda^* = 1 - \frac{1}{r}$  is stable for  $-1 < (2-r) < 1$  (i.e., for  $1 < r < 3$ ) & is unstable for  $r > 3$

- the critical slope  $f'(x^*) = -1$  is attained when  $r=3$ ; this is a flip bifurcation; it is often associated w/ period-doubling & indeed spawns a 2-cycle here

- a 2-cycle exists iff  $\exists p, \varepsilon \text{ s.t. } f(p) = \varepsilon, f(\varepsilon) = p$  ( $\text{if } p \neq \varepsilon$ )  $\Leftrightarrow f(f(p)) = p$  with  $f(p) \neq p$

(here,  $f(x) = r x (1-x)$ )  $\uparrow$  so  $p$  is a fixed pt. of the 2nd-iterate map  $f^2$

- need to find  $f^2(x) - x = 0$

$$\Rightarrow r^2 x (1-x)[1 - r x (1-x)] - x = 0$$

- $x, x - (1 - \frac{1}{r})$  are necessarily 2 of the roots because fixed points  $x^*$  automatically satisfy  $f^2(x^*) = x^*$

$p, \varepsilon \in \mathbb{R}$  for  
 $r \in 3, \text{ so no}$   
 $\curvearrowright$  2-cycle  
 there

$\Rightarrow$  quadratic whose roots are

$$p, \varepsilon = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

which are real for  $r >$

at  $r=3$ , the point coincides w/  $x^* = 1 - \frac{1}{r} = \frac{2}{3}$  so it bifurcates from there

$\therefore$  if a 2-cycle  $\forall r > 3$

# Liapunov exponent (discrete version)

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- we used the word "chaotic" but for that name to be apt, we need sensitive dependence on initial conditions, and we haven't yet shown any such thing
- intuition: given  $x_0$ , consider a nearby point  $x_0 + \delta_0$  ( $\delta_0$  is a small separation); let  $d_n$  be the separation after  $n$  iterates; if  $|d_n| \approx |\delta_0| e^{n\lambda}$ , then we call  $\lambda$  the Liapunov exponent;  $\lambda > 0$  is a signature of chaos (though, note, not a definition of it) because it means nearby points separate exponentially fast

take logs

- $\lambda \approx \frac{1}{n} \ln \left| \frac{d_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$

$$\rightarrow \frac{1}{n} \ln |(f^n)'(x_0)| \text{ as } \delta_0 \rightarrow 0$$

- now,  $(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$

$$\Rightarrow \lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- taking  $n \rightarrow \infty$  limit, we get the definition

$$\lambda(x_0) := \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right\}$$

↑  
depends on  $i$ .

(but it is the same  $\forall x_0$  in the basin of the same "attractor")

- $\lambda < 0$  for stable fixed pts.  
•  $\lambda > 0$  for chaotic attractor

"attractor" =  
a subset of  
space to  
which trajectory →  
go as  $t \rightarrow \infty$ ; basin  
of attraction = set  
of pts. not go there

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- "universality": plot an orbit diagram of  $x_{n+1} = r \sin \pi x_n$ , for  $0 \leq r \leq 1$ ,  $0 \leq x \leq 1$  & compare it to that for logistic map (plot that too); you'll find that their qualitative dynamics are identical (they're different quantitatively — the windows of periodic behavior have different sizes, for example)

- There's a quantitative form of universality as well
  - in fact, if we consider any unimodal map, one gets  $\delta = \lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\dots$  (amazing!)
  - (Feigenbaum 1979; used some ideas from "renormalization" in statistical physics)

Renormalization (will do this intuitively in the lectures; possibly more rigorously in the hw...)

- let  $f(x, r)$  denote a unimodal map that undergoes a "period-doubling route to chaos" as  $r$  increases

- $X_m = \max f$
- $r_n =$  value of  $r$  at which a  $2^n$ -cycle is born
- $R_n =$  value of  $r$  at which the  $2^n$ -cycle becomes superstable

- (e.g.) consider  $f(x, r) = r - x^2$

- at  $R_0$ , the map has a superstable fixed pt. by def.; the fixed pt. condition is  $x^* = R_0 - (x^*)^2$  & the superstability condition is  $\lambda = \left. \frac{\partial f}{\partial x} \right|_{x=x^*} = 0$

34a]

when substituted

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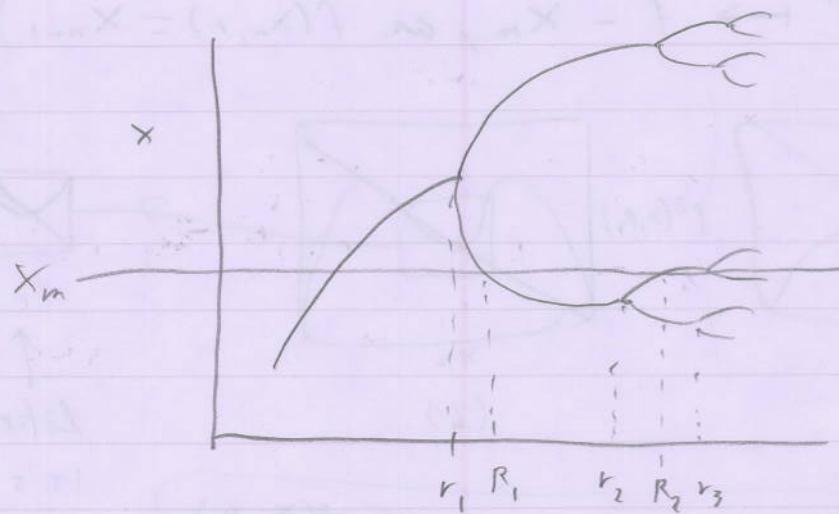
- $\frac{\partial f}{\partial x} = -2x \Rightarrow x^* = 0$ , which gives  $R_0 = 0$  / into fixed pt. condition

at  $R_1$ , the map has a superstable 2-cycle (let  $p_1, p_2$  denote the pts of the cycle)

• Superstability requires  $\lambda = (-2p)(-2s) = 0$ , so  $x = 0$  must be one of the pts.

• period-2 condition  $f^2(0, R_1) = 0 \Rightarrow R_1 - R_1^2 = 0 \Rightarrow R_1 = 1$   
 (the other root gives a fixed pt, not a cycle)

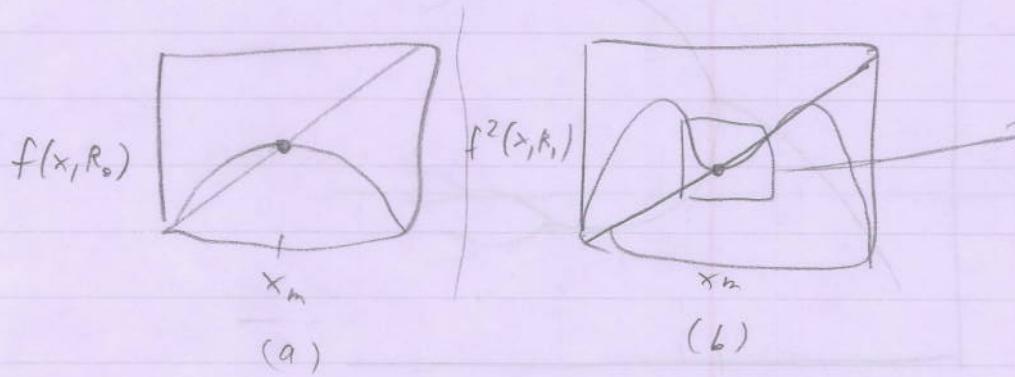
• in general (see this e.g.), a superstable cycle of a unimodal map always contains  $x_m$  as one of its pts  
 $\Rightarrow$  one can find  $R_n$  graphically



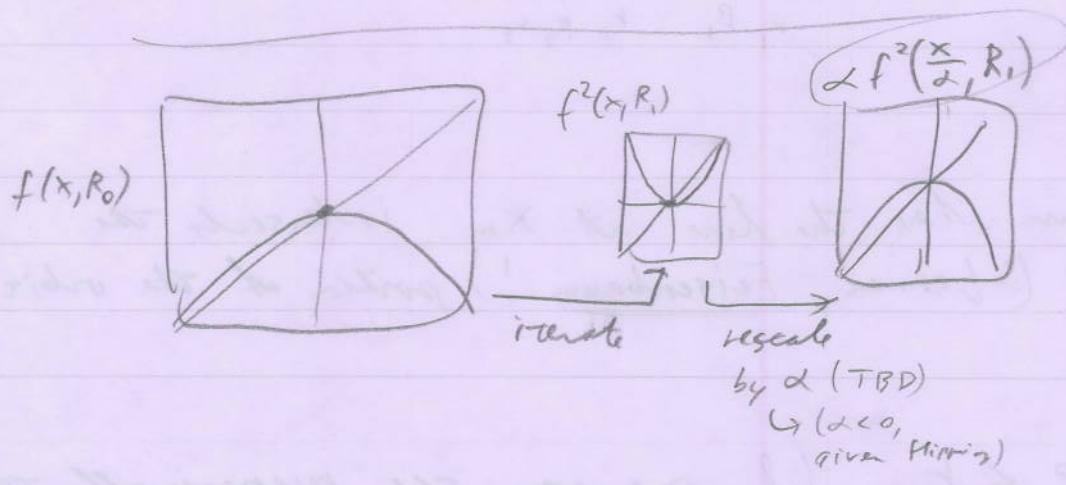
•  $R_n$  occur where the line at  $x_m$  intersects the "figtree" (in Berman, Feigenbaum!) portion of the orbit diagram

• note  $r_n < R_n < r_{n+1}$  (& one can see numerically that the spacing between successive  $R_n$  also shrinks by the universal factor  $\delta \approx 4.669$ )

- The renormalization procedure is based on the self-similarity (fractal nature) of the figure (the twigs look like earlier branches, except that they're smaller)
    - some process occurs over & over - a  $2^n$  cycle is born, then becomes superstable, then loses stability in a period-doubling bif.
  - To express self-similarity mathematically, compare  $f$  w/  $f^2$  at corresponding  $r$  values & then "renormalize" one map into the other ( $f(x, R_0)$  vs.  $f^2(x, R_1)$ )
    - $X_m$  is a superstable pt. of both maps.
  - helpful to map  $x \mapsto x - X_m$  (translate origin to  $X_m$ )  
 $(\Rightarrow f \mapsto f - X_m, \text{ or } f(X_m, r) = X_{m+1})$



like (a), except  
it's smaller &  
inverted



35a) need to blow up fig. by  $\lambda R$  "scale factor"  
 s.t.  $\lambda < 0, |\lambda| > 1$

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$$\Rightarrow f^2(x, R_1) \mapsto \lambda f^2\left(\frac{x}{\lambda}, R_1\right)$$

• find  $\lambda$  with  $f(x, R_0) \approx \lambda f^2\left(\frac{x}{\lambda}, R_1\right)$

$\hookrightarrow f$  has been renormalized by taking its second iterate, rescaling  $x \mapsto \frac{x}{\lambda}$ , & shifting  $R$  to next superstable value

(note: renormalization is far, far more general than this)

• One can continue from here:

•  $f^4$  has a superstable fixed pt. if we shift  $R$  to  $R_2$

$$\circ \text{same reasoning as above} \Rightarrow f^2\left(\frac{x}{\lambda}, R_1\right) \approx \lambda f^4\left(\frac{x}{\lambda^2}, R_2\right)$$

$$\Rightarrow f(x, R_0) \approx \lambda^2 f^4\left(\frac{x}{\lambda^2}, R_2\right)$$

(renormalizing  $n$  times)  $\Rightarrow f(x, R_0) \approx \lambda^n f^{(2^n)}\left(\frac{x}{\lambda^n}, R_n\right)$

• Feigenbaum found numerically that

$$(59) \quad \lim_{n \rightarrow \infty} \lambda^n f^{(2^n)}\left(\frac{x}{\lambda^n}, R_n\right) = g_\circ(x), \text{ where } g_\circ(x)$$

is a universal function of a superstable fixed pt.

• limiting function only exists for a specific  $\lambda = -2.5029\dots$  (another Feigenbaum const.)

- "universal" means that  $g_0$  is (almost) indep of original  $f$
- the explanation comes from the form of (59):

→  $g_0(x)$  depends on  $f$  only through its behavior near  $x=0$ , as that is all that survives in the argument  $\frac{x}{2^n}$  as  $n \rightarrow \infty$ ; w/ each renormalization, we're zooming in on a smaller & smaller neighborhood, so we lose almost all info about the global shape of  $f$

(Note: the order of the max. is never forgotten, so the precise statement is that  $g_0(x)$  is universal  $\forall f$  w/ a quadratic maximum (de generic case); there's a different  $g_0(x)$  for  $f$ 's w/ a 4th-degree max, etc.)

- To obtain other universal functions  $g_i(x)$ , start w/  $f(x, R_i)$  instead of  $f(x, R_0)$ :

$$g_i(x) = \lim_{n \rightarrow \infty} \alpha^n f^{(2^n)}\left(\frac{x}{2^n}, R_{n+i}\right)$$

↑  
universal function w/ a stable  $2^i$ -cycle

- Most interesting case is when we start w/  $R_i = R_\infty$  (at the onset of chaos), as

$$f(x, R_\infty) \approx \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right)$$

(we don't have to shift  $r$ )

- The limiting function  $g(x) = g_\infty(x)$  satisfies

$$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right) \quad (60)$$

"Feigenbaum-(vicanovic) functional equation"

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- need to specify b.c.'s

(smooth)

- after shift to origin, all unimodal f's have a max at  $x=0 \Rightarrow g'(0)=0$

- can set  $g(0)=1$  wlog (this just defines a scale for x; if  $g(x)$  is a sol. for (60), then so is  $\mu g(\frac{x}{\mu})$ , w/ the same d)

- Now solve for  $g(x)$  & d:

- at  $x=0$ ,  $g(0)=\lambda g(g(0))$ , but  $g(0)=1$ , so  $1=\lambda g(1)$   
 $\Rightarrow \lambda = \frac{1}{g(1)}$ , so  $\lambda$  is determined by  $g(x)$

- no known closed form sol. for  $g(x)$  exists (open problem... good luck), so we do a power series

$$g(x) = 1 + c_2 x^2 + c_4 x^4$$

( $c_i \neq 0$  assume the max is quadratic)

- find coeffs by substituting into (60) & matching like powers of x

(one finds  $c_2 \approx -1.5276$ ,  $c_4 \approx 0.1048$ ,  $\lambda \approx -2.5028$ )  
(Freigebaum, 1979)

- Van also get the value of  $\delta$ , but we need more sophisticated arguments for that (involving operators in function spaces, Frechet derivatives, etc.)

- (in hw, you'll do an approx. calculation)

## 6b] . 5. 2D maps

as we've seen

- one can use maps to gain an intuition for what happens in systems of DEs (flows); e.g., one can obtain a map by taking a Poincaré section & the chaotic<sup>(or order)</sup> properties of such a map (1D, 2D, etc.) can shed light on chaotic (or other properties) of the original flow

- e.g., forced pendulum numerics in hor 2

- some 'relatively simple' 2D maps (that exhibit very rich behavior!) can be particularly helpful to develop intuition about chaos, chaotic attractors, strange attractors, etc.

roadmap → [ • we will now define these concepts a bit better, study some 2D maps, & then get back to chaos in the DEs

- there is no universal def. of chaos (so if you look around, you'll find lots of variant), but there are some common ingredients

$\nearrow$  ( $t \rightarrow \infty$  in most defn.)

- def: chaos is aperiodic, long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions

- aperiodic long-term behavior  $\Rightarrow \exists$  trajectories that do not settle down to equilibria, periodic orbits, or quasi-periodic orbits as  $t \rightarrow \infty$ ; (in practice, one requires that such trajectories are not too rare - might require an open set of them or that they don't have measure 0, etc.); in non-conservative systems, they might live on a strange attractor (defined below) as  $t \rightarrow \infty$  & in Hamiltonian systems (or a stable manifold, if it's  $t \rightarrow -\infty$ )  $\nearrow$  just gets lots of mixing as the trajectory moves

- deterministic  $\Rightarrow$  system has no random or noisy inputs or parameters  
 (irregular behavior comes from the system's nonlinearity & not from some other source)

- sensitive dependence on initial conditions (colloquially, this is called the 'butterfly effect') means that nearby trajectories separate exponentially fast, which you can measure by finding positive Lyapunov exponents (the version defined for vector fields) in parts of phase space

$\hookrightarrow$  mixing of phase space (a chaotic family of trajectories will visit lots of phase space; this notion can be made rigorous & is often used to define chaos, especially in Hamiltonian systems and for applications to fluid mechanics)

- attractor = a set to which all neighboring trajectories converge (e.g., stable equilibria & stable limit cycles); a repeller is just like an attractor except for  $t \rightarrow -\infty$  instead of  $t \rightarrow \infty$

- def: an attractor is a closed set  $A$  with the following properties:

- (1)  $A$  is an invariant set: any trajectory  $\tilde{x}(t)$  that starts in  $A$  stays in  $A$  for all time

7b]

- (2) A attracts an open set of conditions:  $\exists$  an open set  $V$  containing  $A$  s.t. if  $\tilde{x}(0) \in V$ , then the distance from  $\tilde{x}(t)$  to  $A$  tends to 0 as  $t \rightarrow \infty$  (this means that  $A$  attracts all trajectories that start sufficiently close to it); the largest such  $V$  is called the basin of attraction of  $A$
- (3)  $A$  is minimal:  $\nexists$  a proper subset of  $A$  that satisfies conditions (1) & (2)

def.]

- a chaotic attractor is an attractor that exhibits sensitive dependence on initial conditions (positive Liapunov exp.)
- a "strange" attractor is an attractor that consists of something other than the equilibria, periodic orbits, quasiperiodic orbits, or the union of such objects; they often have a fractal character (which is why they're called strange) & are usually but not always also chaotic ( $\exists$  strange, non-chaotic attractors)
- note: many books state incorrectly that strange attractors & chaotic attractors are the same thing
- Anyway, now let's do some 2D maps, where you'll see some interesting phenomena ...

38a

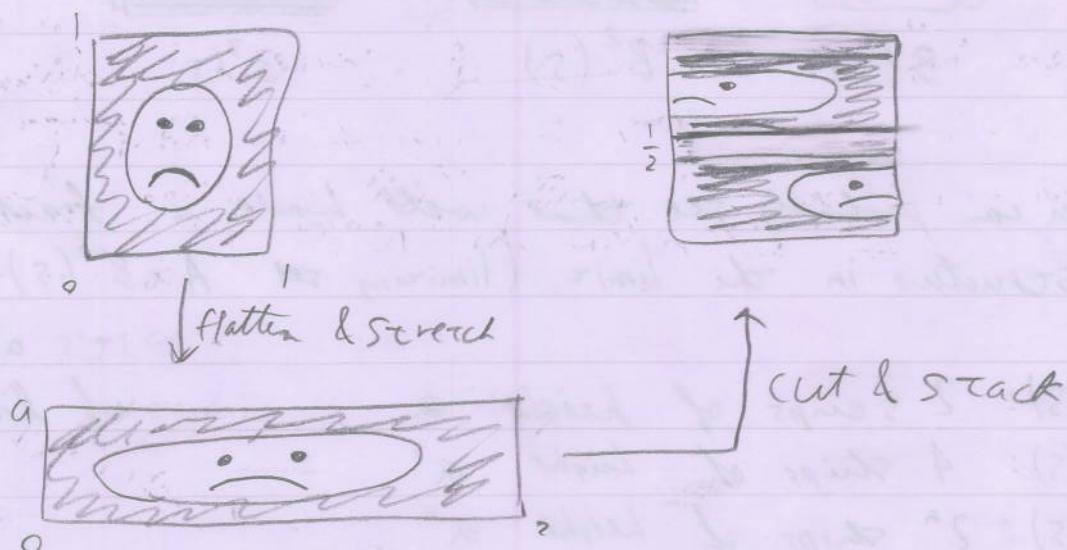
26

MAP  
1/23/10  
BB

- e.g.) Baker's map  $B$  on square  $0 \leq x \leq 1$   
 $0 \leq y \leq 1$

$$B: (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n), & 0 \leq x_n \leq \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}), & \frac{1}{2} \leq x_n \leq 1, \end{cases} \quad (61)$$

where  $a \in (0, \frac{1}{2}]$

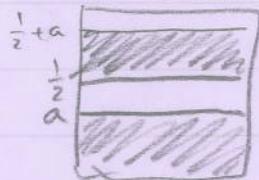
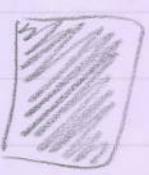


- $B$  is a product of 2 simpler transformations (illustrated above)
  - square is flattened & stretched into a  $2 \times a$  rect.
  - rect is cut in half ( $\rightarrow$  two  $1 \times a$  recs.) & the right half is stacked on top of the left at its base at  $y = \frac{1}{2}$
- $B$  exhibits sensitive dependence on initial conditions / it has uncountably many chaotic orbits
- will show that for  $a < \frac{1}{2}$ ,  $B$  has a strange attractor A that attracts all orbits

386

- more precisely, will show that  $\exists$  a set  $A$  s.t.  $\forall$  i.c.  $(x_0, y_0)$ , the distance from  $B^n(x_0, y_0)$  to  $A$  goes to 0 as  $n \rightarrow \infty$

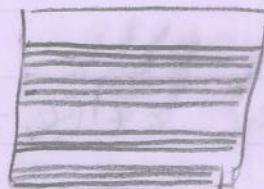
- first, construct the attractor;  $S = \{(x, y) : x \in [0, 1], y \in [0, 1]\} = \text{unit square}$



$B(S)$



$B^2(S)$



$B^3(S)$

5

- you can probably see that will have a fractal structure in the limit (limiting set  $A = B^\infty(S)$  is a fractal)

↳ it's a Cantor set

•  $B(S)$ : 2 strips of height  $a$

of line segments

•  $B^2(S)$ : 4 strips of height  $a^2$

•  $B^n(S)$ :  $2^n$  strips of height  $a^n$

- to be sure there is a "limiting set", we invoke a standard thm. from point-set topology

↳  
the successive images of the square are nested inside each other:  $B^{n+1}(S) \subset B^n(S) \quad \forall n$

↳ each  $B^n(S)$  is a compact set

↳ Then  $\Rightarrow$  the countable intersection of a nested family of compact sets is a non-empty compact set  $A$  and  $A \subset B^n \quad \forall n$

- We also use the nesting property to show that  $A$  attracts all orbits; the pt.  $B^n(x_0, y_0)$  lies somewhere in one of the strips of  $B^n(S)$ , and all pts. in these strips are within a distance  $a^n$  of  $A$  (because  $A \subset B^n(S)$ ); because  $a^n \rightarrow 0$  as  $n \rightarrow \infty$ , the distance from  $B^n(x_0, y_0)$  to  $A$  goes to 0 as  $n \rightarrow \infty$  (as required) ✓

- $a < \frac{1}{2} \Rightarrow B$  shrinks area in phase space ( $\text{area}(B(R)) < \text{area}(R)$ )

↑  
for an attractor,  
need strict inequality

(this happens for attractors in flows as well)

- ⇒  $A$  must have  $\text{area} = 0$ ,  $B$  can't have any repelling fixed pts. (such a pt. would expand area elements in its neighborhood)

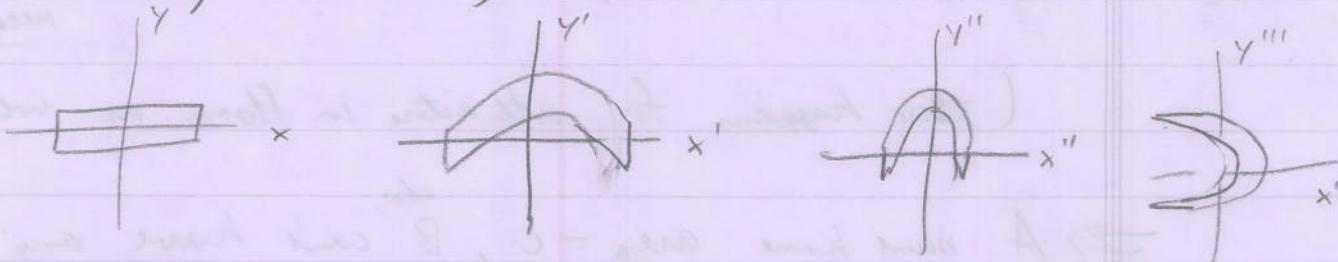
- when  $a = \frac{1}{2}$ ,  $B$  is area-preserving ( $\text{area}(B(R)) = \text{area}(R)$ ), now  $S$  is mapped onto itself (no gaps between strips) &  $B$  has qualitatively different dynamics in this case (there is still chaotic dynamics, but things don't settle down to an attractor)
- note: a map or flow that contracts "volume" ( $\text{area} = 2 - \text{vol}$ ) in phase space is called dissipative

396] e.g.) Hénon map

- historically, invented to capture the essential features of the Lorenz system (which we'll discuss soon) of ODEs

$$\begin{cases} X_{n+1} = Y_n + 1 - aX_n^2 \\ Y_{n+1} = bX_n \end{cases} \quad a, b \in \mathbb{R} \quad (62)$$

- Hénon wanted the following chain of transformations (to simulate the stretching and folding that occurs in the Lorenz system):



- given by the following sequence of transformations:

$$T^1: \begin{cases} x' = x \\ y' = 1 + y - ax^2 \end{cases}$$

The composite transformation

$$T = T''' \circ T'' \circ T'$$

$$T^2: \begin{cases} x'' = bx \\ y'' = y' \end{cases}$$

yields

$$T^3: \begin{cases} x''' = y'' \\ y''' = x'' \end{cases}$$

(62), where  $(x_n, y_n) \equiv (x, y)$

and  $(x_{n+1}, y_{n+1}) \equiv (x''', y''')$

(Elementary)

- Properties of T:

(1) T is invertible (so each pt. has a unique past)

(2) T is dissipative

(3) for certain parameter values, T has a trapping region  
(i.e., ∃ a region R that gets mapped inside itself)

(4) Some trajectories of  $T$  escape to infinity

- To show  $T$  is invertible, we just do some algebra to

$$\begin{aligned} \text{get: } T^{-1} & \left\{ \begin{array}{l} x_n = b^{-1} y_{n+1} \\ y_n = x_{n+1} - 1 + ab^{-2}(y_{n+1})^2 \end{array} \right. \quad (63) \end{aligned}$$

$\Rightarrow$  invertible as long as  $b \neq 0$

- We'll show that  $T$  is area-contracting for  $b \in (-1, 1)$

To determine whether a 2D map  $\begin{cases} X_{n+1} = f(x_n, y_n) \\ Y_{n+1} = g(x_n, y_n) \end{cases}$

is area-contracting, we compute the determinant of the Jacobian  $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$  & show  $|\det(J)| < 1 \quad \forall (x, y) \in \mathbb{R}^2$

$$\text{for Hénon, } J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} \Rightarrow \det J = -b \quad \forall (x, y)$$

$\therefore$  contracting for  $b \in (-1, 1)$

- I'll let you verify properties (3, 4) on your own

- One still has to choose parameters carefully to get a nice attractor; try  $a=1.4, b=0.3$

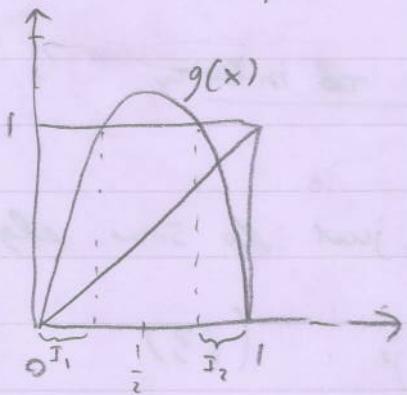
### Symbolic Dynamics

this map again

$$g: \mathbb{R} \rightarrow \mathbb{R}, \quad g(x) = ax(1-x) \quad (64)$$

$(a > 2 + \sqrt{5})$

ob.  $g$  has a fixed pt. at  $0$ ; also,  $g(1) = 0$



$$I_1 = \left[ 0, \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{a}} \right]$$

$$I_2 = \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{a}}, 1 \right]$$

$$I = [0, 1]$$

- $a > 2 + \sqrt{5} \Rightarrow \exists \lambda > 1$  s.t.  $|g'(x)| \geq \lambda$  on  $g^{-1}(I) \cap I$

- remark: largest possible value of  $\lambda$  is  $a \sqrt{1 - \frac{4}{a}}$

- the orbit  $\{g^i(x)\}_{i=0}^\infty$  of almost all pts  $x \in I$  eventually escape from  $I$  and tend toward  $-\infty$ , but  $\exists$  an invariant set of points  $\Lambda$  whose iterates remain in  $I$ ; we'll use symbolic dynamics to describe this set

- partition  $I \cap g^{-1}(I)$  into  $I_1$  &  $I_2$

- associate to each  $x \in \Lambda$  a sequence  $\{a_i\}_{i=0}^\infty$  of 1's and 2's defined by  $a_i = j$  if  $g^i(x) \in I_j$ ;  
(i.e., we are keeping track of each iterate based on whether it goes to the left or to the right)

- observation: if  $J \subset I$ , then  $g^{-1}(J)$  consists of exactly 2 subintervals (one contained in  $I_1$  & the other contained in  $I_2$ )

- $|J| = \text{length}(J)$  is  $\geq \lambda$  times the length of each component of  $g^{-1}(J)$  because  $|g'(x)| \geq \lambda$  if  $g(x) \in I$

- this gives several consequences:

(1) every symbol sequence is associated to some point of  $\Lambda$

• [to prove use finite intersection property of finite sets, & show that  $I_{a_0, \dots, a_n} = I_{a_0} \cap g^{-1}(I_{a_1} \cap g^{-1}(I_{a_2} \cap \dots \cap I_{a_n}))$  is a closed, nonempty set, which follows from above because  $g^{-1}(I_{a_0, \dots, a_n})$  intersects both  $I_1$  &  $I_2$ ]

(2) Distinct pts. of  $\Lambda$  have distinct associated symbol sequences

[follows from fact that  $|I_{a_0, \dots, a_n}| < \lambda^{-(n+1)}$ ]

(3)  $\Lambda$  is a Cantor set (i.e., a closed set that contains no interior pts. or isolated pts.)

to show this, write  $\Lambda_n = \bigcup I_{a_0, \dots, a_n}$ , where the union is taken over all  $(n+1)$ -tuples of 1's & 2's; then  $\Lambda_n$  consists of  $2^{n+1}$  closed subintervals, and each component of  $\Lambda_{n-1}$  contains exactly 2 components of  $\Lambda_n$ ; because the lengths of the components of  $\Lambda_n$  go to 0 w/ increasing  $n$ ,  $\Lambda = \bigcap_{n \geq 0} \Lambda_n$  is a Cantor set

(4) the sequence associated to  $g(x)$  is obtained from the sequence associated to  $x$  by dropping the first term

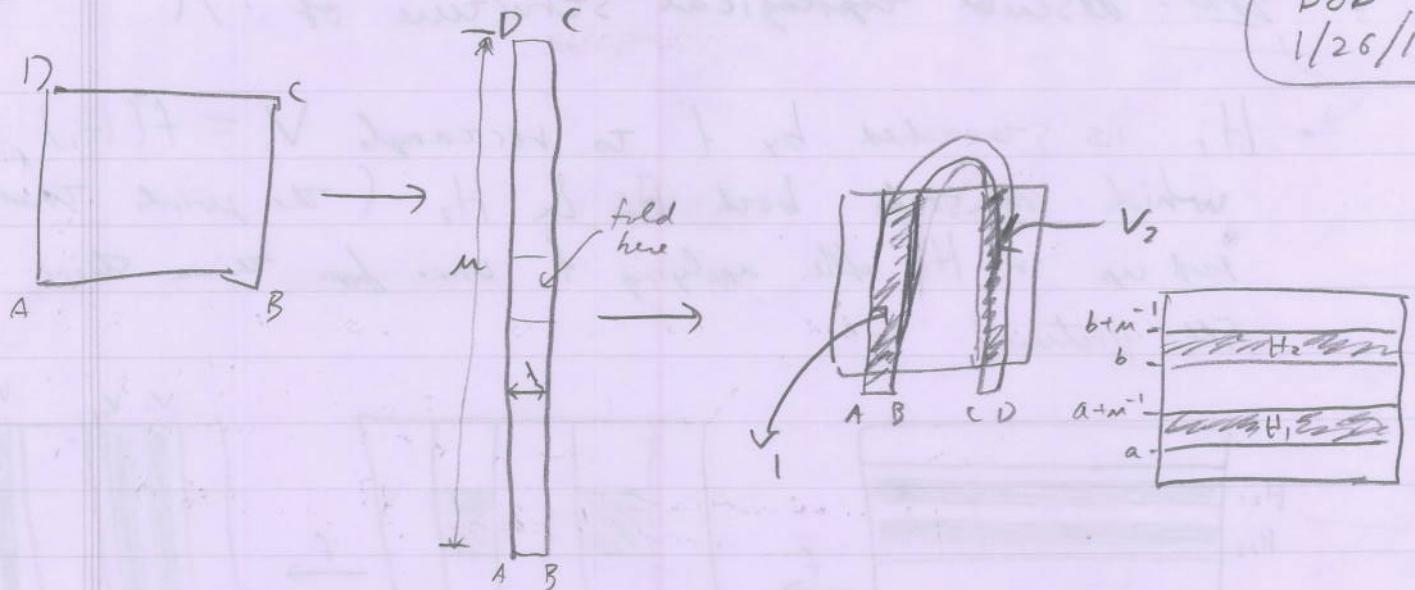
716] ∵ we can label each  $x \in A$  uniquely by a semi-infinite sequence  $\phi(x) = \{a_i(x)\}_{i=0}^{\infty}$ , where  $a_i$  is 1 or 2; the  $a_i$ 's are chosen to reflect the dynamics (i.e., the orbit structure) of  $f$   
(semi-infinite because we're thinking of forward map)

- note that a countable  $\infty$  of periodic orbits & asymptotically periodic orbits (how would you see this using symbolic dynamics?)

### Smale horseshoe

- one of the motivating examples in the modern theory of dynamical systems (the map  $g$  is related to it)
- it's described in terms of an invertible planar map that can be thought of as a Poincaré map arising from a 3-dim autonomous dynamical system or a forced oscillator problem (indeed, we'll see how horseshoes can show up in chaotic dynamics of such DDEs; e.g., arises whenever one has transverse homoclinic orbits, as in forced pendulum)
- $S = [0, 1] \times [0, 1] \subset \mathbb{R}^2$  is unit square
- map  $f: S \rightarrow \mathbb{R}^2$  s.t.  $f(S) \cap S$  consists of 2 components as shown in picture:

42a]

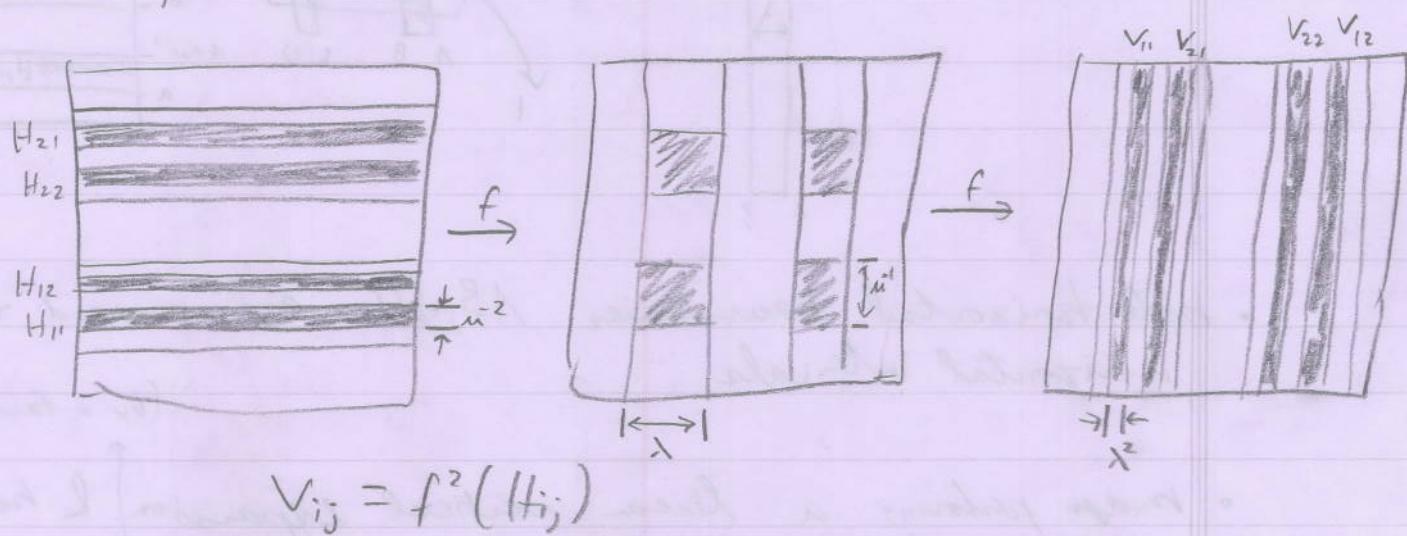
 MPP  
 B&B  
 1/26/10


- note: horizontal boundaries  $AB, DC$  are mapped to horizontal intervals (by a factor of  $m$ )
- map performs a linear vertical expansion & horizontal contraction (by a factor of  $\lambda$ ) of  $S$   
 (the folded portion is outside  $S$ , so restricted to  $S \cap f^{-1}(S)$ , the map is linear)
- one can see that  $f^{-1}(S \cap f(S)) = S \cap f^{-1}(S)$  is two horizontal bands  $H_1 = [0, 1] \times [a, a+m^{-1}]$  and  $H_2 = [0, 1] \times [b, b+m^{-1}]$ , on which  $f$  has const. Jacobian  $J = \begin{pmatrix} \pm\lambda & 0 \\ 0 & \pm m \end{pmatrix}$  (+ on  $H_1$ , - on  $H_2$ ), w/  $0 < \lambda < \frac{1}{2}$  &  $m > 2$  (on each  $H_i$ ,  $f$  compresses horizontal segments by a factor of  $\lambda$  & stretches vertical segments by a factor of  $m$ )
- as  $f$  is iterated, most points either leave  $S$  or are not contained in an image  $f^i(S)$ ; those that do remain  $\forall$  time form a set  $\Lambda = \{x | f^i(x) \in S, -\infty < i < \infty\}$

42b]

- Goal: describe topological structure of  $\Lambda$

- $H_i$  is stretched by  $f$  to rectangle  $V_i = f(H_i)$ , which intersects both  $H_1$  &  $H_2$  (the points that end up in  $H_i$  after applying  $f$  come from three strips; see picture)



- $H_1 \cup H_2 = f^{-1}(S \cap f(S))$ , so the four thin strips constitute  $f^{-2}(S \cap f(S) \cap f^2(S))$

- continue inductively to get that  $f^{-n}(S \cap f(S) \cap \dots \cap f^n(S))$  is the union of  $2^n$  horizontal strips; each of these has thickness  $n^{-n}$  because  $|\frac{\partial f}{\partial y}| = n$  at all pts. in  $H_1 \cup H_2$  & the first  $(n-1)$  iterates of the horizontal strips remain inside  $H_1 \cup H_2$ ; the intersection of all these strips as  $n \rightarrow \infty$  forms a Cantor set of horizontal segments

43a)

 MAP  
 B8b  
 1/27/10

- now consider image under  $f^n$  of one of the  $2^n$  horizontal strips in  $f^{-n}(S \cap f(s) \cap \dots \cap f^n(s))$

- chain rule  $\Rightarrow D[f^n] = \begin{pmatrix} \pm \lambda^n & 0 \\ 0 & \pm \mu^n \end{pmatrix}$   
 $\uparrow$   
 Jacobian matrix of  $f^n$

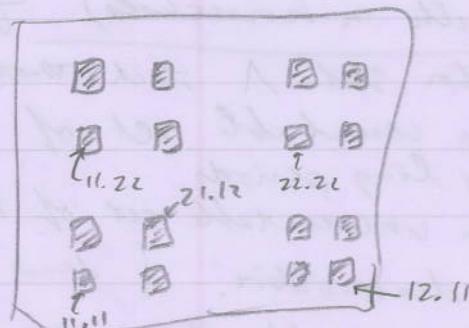
$\Rightarrow$  image is a rectangle of horizontal width  $\lambda^n$  that extends from top to bottom of square ( $f^n$  is injective (1-1), so all the bars are distinct)

$\therefore S \cap f(s) \cap \dots \cap f^n(s)$  is the union of  $2^n$  vertical strips, each of width  $\lambda^n$ ; the intersection of these sets  $\forall n \geq 0$  gives a Cantor set of vertical segments composed of those pts. that are in the images of all the  $f^n$

- to be in  $\Lambda$ , a pt. must be in both of the 2 Cantor sets above (vertical & horizontal)
- $\therefore \Lambda$  is itself a Cantor set; its components are each pre. & each pt. of  $\Lambda$  is an accumulation pt. for  $\Lambda$

e.g. 1

$$f^{-2}(s) \cap f^{-1}(s) \cap S \cap f(s) \cap f^2(s)$$

 $a_2 a_1 - a_1 a_2$

- symbolic dynamics will be helpful for characterizing  $\Lambda$
  - each pt.  $x \in \Lambda$  is characterized by a bi-infinite sequence  
(Because the map is invertible, we go in both directions)
- $$\hookrightarrow \vec{a} = \{a_i\}_{i=-\infty}^{\infty} = \dots a_{-2} a_{-1}, a_0 a_1 \dots$$

**Thm:** There is a 1-1 correspondence  $\Phi$  between  $\Lambda$  & the set  $\Sigma$  of bi-infinite sequences of 2 symbols s.t. the sequence  $\vec{b} = \Phi(f(x))$  is obtained from the sequence  $\vec{a} = \Phi(x)$  by shifting indices one place:  $b_i = a_{i+1}$ . The set  $\Sigma$  has a metric defined by

$$d(a, b) = \sum_{i=-\infty}^{\infty} \delta_i 2^{-|i|}, \quad \delta_i = \begin{cases} 0, & \text{if } a_i = b_i \\ 1, & \text{if } a_i \neq b_i \end{cases}$$

The map  $\Phi$  is a homeomorphism from  $\Lambda$  to  $\Sigma$  endowed w/ this metric.

Proof: Exercise in Problem Set 3. (this will help you get comfortable w/ how symbolic dynamics works)

- The description of the horseshoe is 'robust' w.r.t. small changes in  $f$  (this statement can be made mathematically rigorous; it's a "structural stability" theorem), so this type of phenomenon occurs very generally

**Thm.** (Summary of results on horseshoes) The horseshoe map  $f$  has an invariant Cantor set  $\Lambda$  such that:

- $\Lambda$  contains a countable set of periodic orbits of arbitrarily long periods.
- $\Lambda$  contains an uncountable set of bounded nonperiodic motion
- $\Lambda$  contains a dense orbit. ('dense'  $\Rightarrow$  dense = whole set)

Moreover, any sufficiently  $\epsilon$ -close map  $\tilde{f}$  has an invariant Cantor set  $\tilde{\Lambda}$  w/  $\tilde{f}|_{\tilde{\Lambda}}$  topologically equivalent to  $f|_{\Lambda}$ .

[for the sake of consistency, the notation is allowed]