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# Chaos in continuous dynamical systems (flows)

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e.g.)

- Lorenz equations

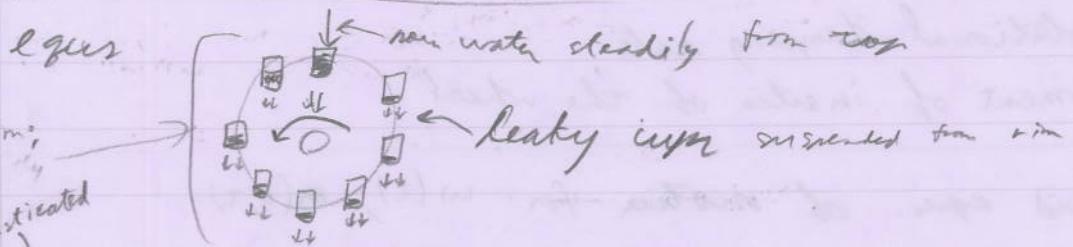
$$\left\{ \begin{array}{l} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{array} \right. \quad (64)$$

parameters  
 $\sigma, r, b > 0$

- Ed Lorenz derived (64) from a drastically simplified model of convection rolls in the atmosphere

- (64) can behave chaotically (it has a chaotic attractor) for many parameter values (Lorenz's computer simulations have turned out to be a watershed moment in science)

- e.g.)
- a chaotic waterwheel: in the 1970s, Willem Malkus & Terence Howard invented a mechanical model of the Lorenz equations

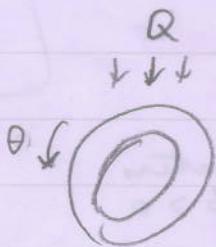


- flow rate too slow  $\Rightarrow$  top cups never fill up enough to overcome friction, so wheel remains motionless
- a little faster & the top cups get heavy enough to start the wheel turning and eventually the wheel settles into steady rotation in 1 direction or the other (which direction depends on i.c.'s; both equally likely by sym.)

- increase the water flow rate more, & we destabilize the steady rotation & the motion becomes chaotic (the wheel switches which direction it rotates in an irregular fashion depending on how much each cup is filled)

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- let's derive some eqns. for this water wheel



- $\theta$  = angle in lab frame (increases counterclockwise)

here are the unknowns  $\begin{cases} \cdot \omega(t) = \text{angular velocity of wheel (increases counterclockwise)} \\ \cdot m(\theta, t) = \text{mass distribution of water around the rim of the wheel, defined s.t. mass between } \theta_1 \text{ & } \theta_2 \text{ is} \end{cases}$

$$M(t) = \int_{\theta_1}^{\theta_2} m(\theta, t) d\theta$$

- $Q(\theta)$  = inflow (rate at which water is pumped in by nozzles above position  $\theta$ )

- $r$  = radius of wheel

- $K$  = leakage rate

- $V$  = rotational damping rate

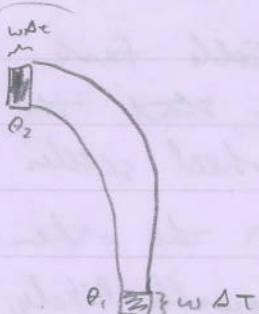
- $I$  = moment of inertia of the wheel

- goal: find eqns. of motion for  $\omega(t)$ ,  $m(\theta, t)$

- conservation of mass (using a standard argument that maybe you saw in fluids)

- consider a sector  $[\theta_1, \theta_2]$

↓  
has mass  $M(t) = \int_{\theta_1}^{\theta_2} m(\theta, t) d\theta$



after an infinitesimal time  $\Delta t$ , there are 4 contributions to  $\Delta M$ :

$$(1) \text{ mass pumped in by nozzles} = \left[ \int_{\theta_1}^{\theta_2} Q d\theta \right] \Delta t$$

$$(2) \text{ mass that leaks out is } \left[ - \int_{\theta_1}^{\theta_2} K m d\theta \right] \Delta t$$

(3) as the wheel rotates, it carries a new block of water into observation sector; this block has mass  $m(A_1)w\Delta t$  (because it has angular width  $w\Delta t$  & mass  $m(A_1)$  per unit angle)

this factor implies that leakage occurs at a rate proportional to the mass of water in the chamber (more water  $\Rightarrow$  a large pressure head  $\Rightarrow$  faster leakage)

[better + more complicated models are conceivable, but this already has good agreement w/ experiments because of the way they were designed]

(4) similarly to (3), the mass carried out of the sector is  $-m(A_2)w\Delta t$

$$\therefore (\text{combining (1) - (4)}): \Delta M = \Delta t \left[ \int_{\theta_1}^{\theta_2} Q d\theta - \int_{\theta_1}^{\theta_2} K m d\theta \right] + m(A_1)w\Delta t - m(A_2)w\Delta t \quad (6)$$

- $m_1(\theta_1) - m_2(\theta_2) = - \int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta} d\theta$

- then divide by  $\Delta t$  & let  $\Delta t \rightarrow 0$

$$\Rightarrow \frac{dM}{dt} = \int_{\theta_1}^{\theta_2} \left( Q - Km - w \frac{\partial m}{\partial \theta} \right) d\theta = \int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial t} d\theta \quad \text{by def. of } M$$

- this holds for all  $\theta_1$  &  $\theta_2$ , so we get

$$\frac{\partial m}{\partial t} = Q - Km - w \frac{\partial m}{\partial \theta} \quad (6)$$

$\uparrow$  this PDE is a continuity eqn.

- still need: an eqn for how  $w(t)$  evolves

- torque balance: rotation of wheel governed by  $F=ma$  (of course), which is expressed as a balance between applied torque & rate of change of angular momentum

- $I :=$  moment of inertia of wheel

•  $I = I(t)$ , but one can show (take some argumentation) that  $I(t) \rightarrow \text{const.}$  as  $t \rightarrow \infty$

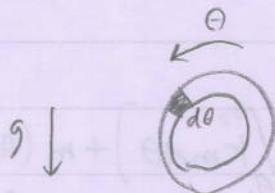
$\Rightarrow$  after transient decay, eqn of motion is

$$I\ddot{\omega} = \underbrace{\text{damping torque} + \text{gravitational torque}}$$

2 sources: (1) Viscous damping from heavy oil in the tank (in the more sophisticated machine)

- both produce torque  $\propto \omega$ ,

$$\text{so damping torque} = -\nu\omega \quad (\nu > 0)$$



(2) "inertial" damping caused by spin-up effect — the water enters the wheel at 0 ang. velocity but is spun up to ang. velocity  $\omega$  before it leaks out

- gravitational torque is like that in an inverted pendulum, (because water is poured in from the top of the wheel)

- in a sector  $d\theta$ , the mass is  $dM = m d\theta$

$$\Rightarrow \text{torque is } d\tau = (dM)gr\sin\theta = mgr\sin\theta d\theta$$

(check sign: when  $\sin\theta > 0$ , torque tends to increase  $\omega$ )

$g = \text{effective gravitational const.} = g_0 \sin\alpha$ , where  $\alpha = \text{tilt of wheel from horizontal}$

- integrating over all elements  $\Rightarrow$

$$\text{gravitational torque} = gr \int_0^{2\pi} m(\theta, t) \sin\theta d\theta$$

∴ torque balance eqn is

$$I\dot{\omega} = -\nu\omega + gr \int_0^{2\pi} m(\theta, t) \sin(\theta) d\theta \quad (67)$$

integro-differential eqn

• (66, 67) completely specify the evolution of the system

- there's some hope in gaining understanding if we can extract a simpler description

•  $m(\theta, t)$  is periodic in  $\theta$ , so expand in a Fourier series

$$m(\theta, t) = \sum_{n=0}^{\infty} [a_n(t) \sin(n\theta) + b_n(t) \cos(n\theta)] \quad \& \text{ substitute into (66, 67)}$$

$\Rightarrow$  amplitude eqns for  $a_n(t), b_n(t)$ ; these are ODEs for the different harmonics or modes of the system

- we'll also need to write the inflow as a Fourier

$$\text{series: } Q(\theta) = \sum_{n=0}^{\infty} q_n \cos(n\theta)$$

$\left\{ \begin{array}{l} \text{no } \sin(n\theta) \text{ terms because} \\ \text{same inflow occurs at } \theta \& -\theta; \\ \text{water is added symmetrically at} \\ \text{the top of the wheel} \end{array} \right.$

(substitute into (66))

$$\Rightarrow \frac{\partial}{\partial t} \left[ \sum_{n=0}^{\infty} a_n(t) \sin(n\theta) + b_n(t) \cos(n\theta) \right] = -\omega \frac{\partial}{\partial \theta} \left[ \sum_{n=0}^{\infty} a_n(t) \sin(n\theta) + b_n(t) \cos(n\theta) \right] + \sum_{n=0}^{\infty} q_n \cos(n\theta) - k \left[ \sum_{n=0}^{\infty} a_n(t) \sin(n\theta) + b_n(t) \cos(n\theta) \right]$$

- differentiate both sides & collect terms; by ~~set~~  
orthogonality of  $\sin(n\theta), \cos(n\theta)$ , we can equate the coeffs for each harmonic separately

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- e.g. coeff of  $\sin(n\theta)$  on LHS is  $a_n$  & on RHS is  $n\omega b_n - K a_n$ , so we get  $\dot{a}_n = n\omega b_n - K a_n \quad (68)$   
 $\text{equating} \quad (n=0, 1, 2, \dots)$

- similarly, [coeff of  $\cos(n\theta)$ ] gives  $\dot{b}_n = -n\omega a_n - K b_n + g_a \quad (69)$

- now rewrite (67) in terms of Fourier series:

$$I \ddot{\omega} = -\nu \omega + g_r \int_0^{2\pi} \left[ \sum_{n=0}^{\infty} a_n(t) \sin(n\theta) + b_n(t) \cos(n\theta) \right] \sin \theta d\theta$$

$$= -\nu \omega + g_r \int_0^{2\pi} a_1 \sin^2 \theta d\theta \quad (\text{all others are zero because of orthogonality})$$

$$= -\nu \omega + \pi g_r a_1 \quad (70)$$

∴ (68, 69, 70) give a closed system for  $a_1, b_1, \omega$  (then three var. have decoupled from  $a_n, b_n$  for  $n \neq 1$ )

$$\Rightarrow \begin{cases} \dot{a}_1 = \omega b_1 - K a_1 \\ \dot{b}_1 = -\omega a_1 - K b_1 + g_a \\ \ddot{\omega} = (-\nu \omega + \pi g_r a_1)/I \end{cases} \quad (71)$$

but w/  
different  
notation

$\uparrow$   
 this is the same as the Lorenz  
eqn. (as long as we're only interested  
in the primary mode!)

- note: one can show that if  $Q(\theta) = g, \cos \theta$ , then for all  $n \neq 1$ ,  $a_n, b_n \rightarrow 0$  as  $t \rightarrow \infty$  (so leading mode dominates after transient dissipates)

- much more difficult/interesting for general  $Q(\theta)$   
 (still a research topic, as far as I know)

- ok, now let's examine the properties of the Lorenz eqn. (using the notation in (71) for now)

### equilibria

- they satisfy  $a_1 = \frac{\omega b_1}{K}$

$$\omega a_1 = g_1 - K b_1 \Rightarrow$$

$$a_1 = \frac{\nu \omega}{\pi g r}$$

$\Rightarrow \omega = 0$  or  $b_1 = \frac{K \nu}{\pi g r}$ , which gives 2 kinds of equilib. to consider:

(1) if  $\omega = 0$ , then  $a_1 = 0$  &  $b_1 = \frac{g_1}{K}$

$\therefore (a_1^*, b_1^*, \omega^*) = (0, \frac{g_1}{K}, 0)$ , which corresponds to a state of no rotation: the wheel is at rest, with the inflow balanced by the leakage

(2) if  $\omega \neq 0$ , then we get  $b_1 = \frac{K a_1}{(\omega^2 + K^2)} = \frac{K \nu}{\pi g r}$

(and note that  $K \neq 0$ )

$$\Rightarrow (\omega^*)^2 = \frac{\pi g r g_1}{\nu} - K^2 \quad (72)$$

if RHS of (72) is  $> 0$ , then we get 2 sol.  $\pm \omega^*$  corresponding to steady rotation in the 2 directions.

a dimensionless quantity called the Rayleigh number

then sol exist iff

$$\frac{\pi g r g_1}{K^2 \nu} > 1$$

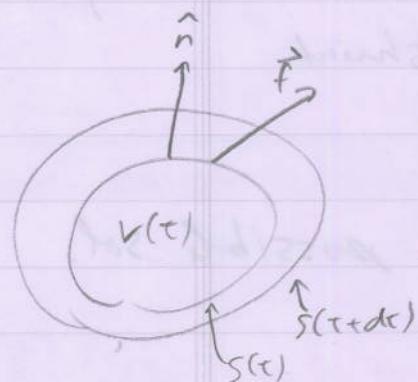
- 47b] • Rayleigh number measures how hard we're driving the system relative to the dissipation  
 (this quantity shows up a lot in fluid mechanics - e.g., in the study of convection)
- now let's go back to the standard notation for the Lorenz eqn & continue from there:
- $$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = r x - y - xz \\ \dot{z} = xy - bz \end{cases}, \sigma, r, b > 0 \quad (73)$$
- nonlinearity
- $r$  = Rayleigh number
  - $\sigma$  = Prandtl number
- when Lorenz was studying (73), he found himself confronted with what seemed like a paradox: he found a range of parameters for which  $\nexists$  stable equilibria or stable limit cycles, yet he had also proved that all trajectories remained confined to a bounded region and were eventually attracted to a set of 0 volume. So what was happening? (The limiting set was in fact a strange, chaotic attractor.)
- Amazing, especially given how innocuous the eqn look.  
 (just 2 nonlinearities)
- Symmetry: if we take  $(x, y) \mapsto (-x, -y)$  in (73), the eqn remains the same
- ∴ if  $(x(t), y(t), z(t))$  is a solution, then so is  $(-x(t), -y(t), z(t))$

- Volume contraction:

- (73) is dissipative: Volumes in phase space contract under the flow

↳ let's look at this for a general 3D dynamical system  $\dot{\vec{x}} = \vec{f}(\vec{x})$ ,  $\vec{x} \in \mathbb{R}^3$

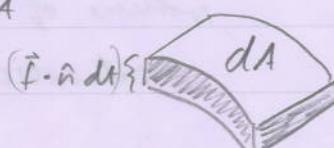
- pick an arbitrary closed surface  $S(t)$  of volume  $V(t)$  in phase space



- think of points on  $S$  as i.c.'s for trajectories & let them evolve for  $dt$ ; then  $S$  evolves to  $S(t+dt)$ . What is  $V(t+dt)$ ?

- let  $\hat{n}$  = outward unit normal on  $S$
- $\vec{f}$  = instantaneous velocity, so  $\vec{f} \cdot \hat{n}$  = outward normal component of velocity

$\Rightarrow$  in time  $dt$ , a patch of area  $dA$  sweeps out a volume  $(\vec{f} \cdot \hat{n} dt)dA$



$$\Rightarrow V(t+dt) = V(t) + [\text{volume swept out by tiny patches of surface, integrated over all patches}]$$

$$= V(t) + \int_S (\vec{f} \cdot \hat{n} dt) dA$$

$$\Rightarrow \dot{V} = \frac{V(t+dt) - V(t)}{dt} = \int_S \vec{f} \cdot \hat{n} dA$$

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Using divergence theorem, we get  $\dot{V} = \int_V \nabla \cdot \vec{f} dV$

- for Lorenz, this gives  $\nabla \cdot \vec{f} = \frac{\partial}{\partial x} [0(y-x)] + \frac{\partial}{\partial y} [rx-y-xz] + \frac{\partial}{\partial z} [xy-bz]$   
 $= -r - 1 - b < 0$

$$\therefore \dot{V} = -(r+1+b)V, \text{ so } V(t) = V(0)e^{-(r+1+b)t}$$

so volume in phase space shrunk exponentially fast

- this impose strong constraints on possible sol.  
to Lorenz eqns

e.g.) the Lorenz eqns. don't have quasiperiodic sol.

Suppose there were; any quasiperiodic sol. would have to lie on the surface of a torus, & the torus would need to be invariant under the flow; hence, the volume of the torus would be const. in time, which contradicts what we've just shown

also, the Lorenz system can't have either repelling equilibria or repelling closed orbits (repelling means that all trajectories are driven away from it); check this yourself

- we'll now look at the equilibria

- The origin  $(x^*, y^*, z^*) = (0, 0, 0)$  is an equilib. for all parameter values
- For  $r > 1$ , we also get a symmetric pair of equilibria:  $x^* = y^* = \pm \sqrt{b(r-1)}$ ,  $z^* = r-1$  (Lorenz called them  $C^+$  &  $C^-$ ); as  $r \rightarrow 1^+$ ,  $C^+$  &  $C^-$  coalesce via  $(0, 0, 0)$  in a pitchfork bifurcation
- linear stability of  $(0, 0, 0)$ :
  - Linearization:  $xy$ ,  $xz$  nonlinearities drop out & the eqn for  $z(t)$  gets decoupled and  $z(t) \rightarrow 0$  exponentially fast as  $t \rightarrow \infty$ , so we can just look at  $x$  &  $y$
  - $\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & 0 \\ r-1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$   
 $\downarrow$   
 has trace  $\tau = -\sigma - 1 < 0$   
 and determinant  $\Delta = \sigma(1-r)$
- $r > 1 \Rightarrow (0, 0, 0)$  is a saddle because  $\Delta < 1$ , but this is a new type of saddle for us because this is a 3D system
  - including  $z$  direction, we have one outgoing dim (1D "unstable manifold") & 2 incoming dim (i.e., 2D stable manifold)
- $r < 1 \Rightarrow$  3D stable manifold (so equilib is a sink)  
 $\hookrightarrow$  this equilib. is a stable node

## 49b] Global Stability of $(0,0,0)$ :

- for  $r < 1$ , we can show that every trajectory approaches  $(0,0,0)$  as  $t \rightarrow \infty$ ; i.e.,  $(0,0,0)$  is globally stable ( $\Rightarrow$  no limit cycles or chaos for  $r < 1$ )
  - to prove this, we'll construct a Lyapunov function — a smooth, positive definite function that decreases along trajectories (see earlier discussion)
  - consider  $V(x,y,z) = \frac{1}{2}x^2 + y^2 + z^2$  ( $V = \text{const. are concentric ellipsoids about } (0,0,0)$ )
    - want to show: if  $r < 1$  &  $(x,y,z) \neq (0,0,0)$ , then  $\dot{V} < 0$  along trajectories; because  $V \geq 0$ ,  $V(\vec{x}(t)) \rightarrow 0$  & hence  $\vec{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$

$$\begin{aligned}\cdot \frac{1}{2} \dot{V} &= \frac{1}{2} x \dot{x} + y \dot{y} + z \dot{z} = (yx - x^2) + (ryx - y^2 - xzy) + (zxy - bz^2) \\ &= (r+1)xy - x^2 - y^2 - bz^2 \\ &= -\left[x - \frac{r+1}{2}y\right]^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - bz^2 \\ &\leq 0 \quad V(x,y,z), \text{ so now we just need to show that } \dot{V} \neq 0. \text{ (for } \vec{x} \neq 0)\end{aligned}$$

$\hookrightarrow \dot{V}=0$  would require each of the 3 terms to vanish simultaneously, which only happens at  $(0,0,0)$

$\therefore (0,0,0)$  is globally stable

## • stability of $C^+$ & $C^-$

- now suppose  $r > 1$ , s.t.  $C^+$  &  $C^-$  exist; one can do the usual eigenvalue computation & show that they are linearly stable for  $1 < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$

(assuming also that  $\sigma-b-1 > 0$ )

• stability is lost via a Hopf bif. at  $r=r_H$

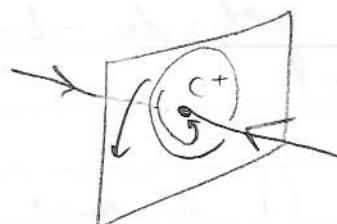
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- this Hopf bif. is subcritical, so there are unstable limit cycles for  $r \in (1, r_H)$

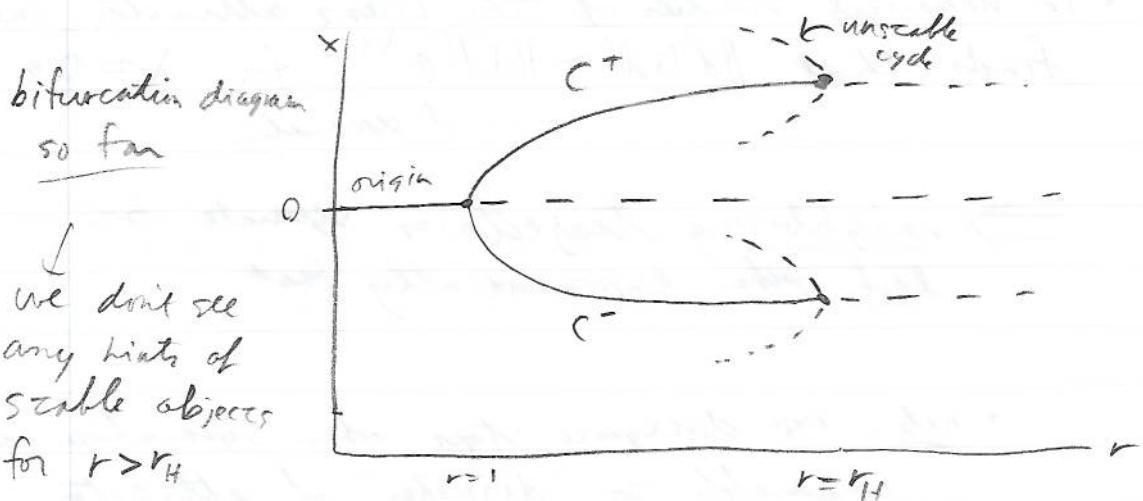
- we get a stable equilib. & a "saddle cycle" (a type of unstable limit cycle that needs  $\geq 3$  dim to occur)



↓  
the cycle has a 2D unstable manifold & a 1D stable manifold

- as  $r \rightarrow r_H^-$ , the cycle shrinks down around the equilib.; at the Hopf bif., the equilib. absorbs the saddle cycle & becomes a saddle pt.; for  $r > r_H$ , there are no attractors in the nhbd.

⇒ for  $r > r_H$ , the trajectories must fly away towards a distant attractor



we don't see any hints of stable objects for  $r > r_H$

but, one can prove that all trajectories eventually enter and remain in a certain large ellipsoid (though as we're in 3D, we can't use any results like Poincaré-Bendixson!); & we don't go to a limit cycle, as Lorenz showed (well, he did it heuristically, but it's now been shown rigorously) that any limit cycle for  $r > r_H$  would have to be unstable (will discuss later)

- 50b]  $\Rightarrow$  any trajectory must have bizarre long-term behavior & in fact what we have here is a strange attractor (that I introduced earlier — and now we're finally going to see it in an ODE)
- look up Lorenz attractor online to see what it looks like!
  - on the Lorenz attractor (& on most strange attractors), we have chaos and hence sensitive dependence on initial conditions (supposing, e.g., that we start on the attractor)
    - $\hookrightarrow$  let's make this more precise:

### Liajanov exponents (continuous version)

- suppose  $\tilde{x}(t)$  is a pt. on the attractor at time  $t$  & consider a nearby pt.  $\tilde{x}(t) + \tilde{\delta}(t)$  ( $\text{small } \|\tilde{\delta}\|$ )
- in numerical studies of the Lorenz attractor, one finds that  $\|\tilde{\delta}(t)\| \sim \|\tilde{\delta}_0\| e^{\lambda t}$  for  $\lambda \approx 0.9$  initial separation

$\Rightarrow$  neighboring trajectories separate from each other exponentially fast

- note: exp. divergence stops when separation is comparable to "diameter" of attractor (it's a saturation effect)
- $\lambda$  is a Liajanov exponent (it's the largest one; there are  $n$  different ones for an  $n$ -dim system & note that they are a function of where you are in phase space)

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- suppose that if a prediction is within a of the true state, we consider it acceptable (so, we're thinking of  $\tilde{x}(t)$  as intended state &  $\tilde{x}(t) + \delta(t)$  an actual state — the difference would be from, e.g., small exp. error that we can't control)  
↳ bad prediction is intolerable when  $\|\delta(t)\| \geq a$ , which occurs after a time  $t_{\text{unstable}} \sim O\left(\frac{1}{\lambda} \ln\left(\frac{a}{\|\delta_0\|}\right)\right)$

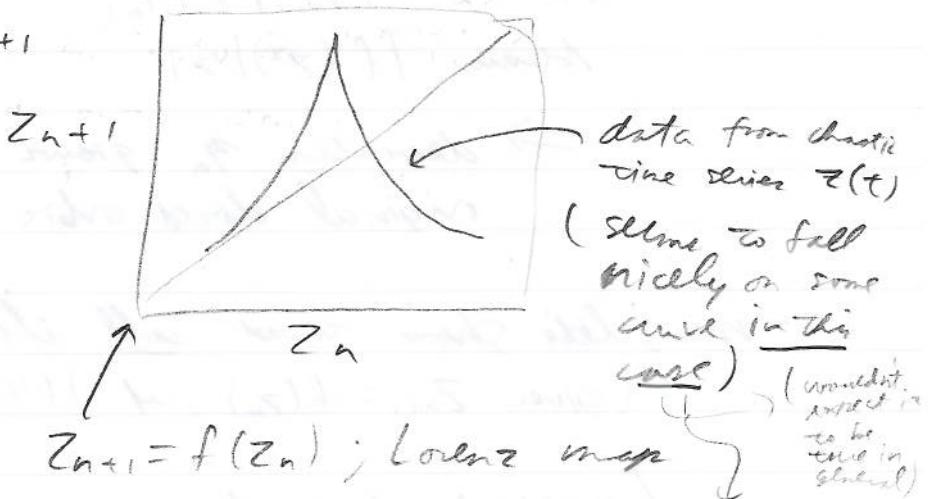
(because of the logarithmic dependence,  $t_{\text{unstable}}$  tends to be very short, we fix any  $\|\delta_0\|$ !)

↳ so chaotic systems do predictability even though they're deterministic

$$\text{e.g. } \|\delta_0\| = 10^{-13}, a = 10^{-3} \Rightarrow t_{\text{unstable}} \approx \frac{10 \ln 10}{\lambda} = \\ (\text{for a } \underline{\text{minuscule}} \text{ initial error!})$$

## Lorenz map

- to study the Lorenz attractor, Lorenz focused on a single 'feature' —  $Z_n$ , the  $n^{\text{th}}$  local maximum of  $\tilde{x}(t)$  (have you looked up a picture of the Lorenz attractor yet?); Lorenz's idea was that  $Z_n$  should predict  $Z_{n+1}$



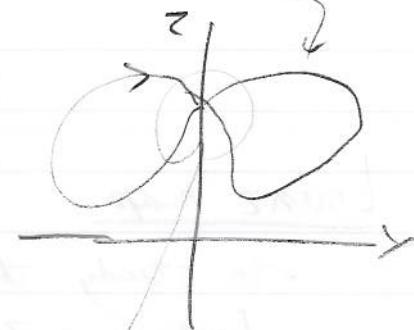
- Lorenz used his map to give a plausible argument to rule out limit cycles (a rigorous proof that the Lorenz attractor exists was done via computer-aided proof (using interval arithmetic) by Warwick Tucker around 2000)

it's not actually a curve because it has some thickness, but it's pretty close (so it can be useful to predict it is)

• Lorenz's plausibility argument:

- take  $Z_{n+1} = f(Z_n)$  to be an actual function (i.e., assume no stickiness in the data); observe that  $|f'(z)| > 1$  everywhere  $\Rightarrow$  any limit cycles that exist must be unstable
- here's why: first consider the fixed pts. of the map  $f$ ; there are pts.  $z^*$  s.t.  $f(z^*) = z^*$   
 $\Rightarrow Z_n = Z_{n+1} = Z_{n+2} = \dots$

from the plot, we see that  $\exists!$  such pt.; it represents a closed orbit like this one:



- To show that this closed orbit is unstable, consider a slightly perturbed trajectory that has  $Z_n = z^* + \gamma_n$  ( $\gamma_n$  small)

- linearization  $\Rightarrow Z_{n+1} \approx f'(z^*)\gamma_n$ ,  
 & we get  $|\gamma_{n+1}| > |\gamma_n|$   
 because  $|f'(z^*)| > 1$

note: not actually an intersection (this is a projection of a 3D system into 2D)

$\Rightarrow$  deviation  $\gamma_n$  grows after each iteration, so the original closed orbit is unstable

- now, let's show that all closed orbits are unstable:  
 (given  $Z_{n+1} = f(Z_n)$ , w/  $|f'(z)| > 1 \forall z$ )

$\hookrightarrow$  think about the sequence  $\{Z_n\}$  corresponding to an arbitrary closed orbit

- it might be complicated but we know it will eventually repeat because the orbit is closed

$\Rightarrow Z_{n+p} = Z_n$  for some integer  $p \geq 1$   
 ( $p$  is the period of the orbit, so  $Z_n$  is periodic)

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- will now show that the corresponding closed orbit is unstable

↳ consider a small deviation  $\eta_n$  & look at it after  $p$  iterations

↳ will show that  $|\eta_{n+p}| > |\eta_n|$ , which implies that the deviation has grown & the closed orbit is unstable

$$\cdot \eta_{n+1} \approx f'(z_n) \eta_n \quad (\text{linearization})$$

$$\begin{aligned} \cdot \eta_{n+2} &\approx f'(z_{n+1}) \eta_{n+1} \\ &\approx f'(z_{n+1}) [f'(z_n) \eta_n] \\ &= [f'(z_{n+1}) f'(z_n)] \eta_n \end{aligned}$$

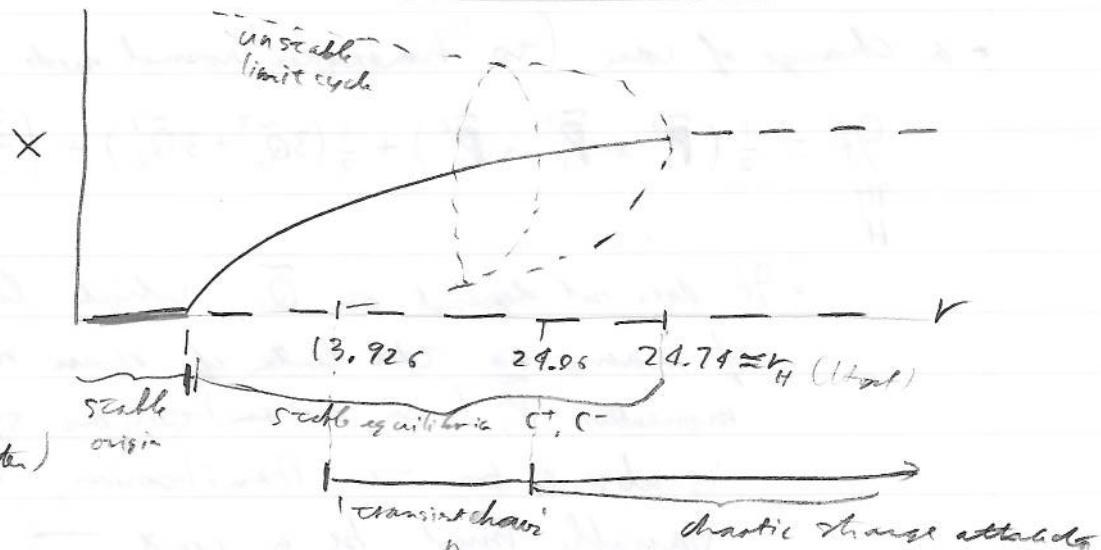
$$\Rightarrow \eta_{n+p} \approx \left[ \prod_{k=0}^{p-1} f'(z_{n+k}) \right] \eta_n \quad (74)$$

• each factor in (74) has absolute value  $> 1$  because  $|f'(z)| > 1 \forall z$

$\Rightarrow |\eta_{n+p}| > |\eta_n|$ , so the closed orbit is unstable

bifurcation diag

$$(a = 10, b = \frac{\delta}{3})$$



(try to duplicate this on the computer)

(try this at home!)

(this just means it seems chaotic for some finite time before settling down to do something else (very difficult to define, given that chaos is defined near  $t \rightarrow \infty$  limit!))

52b

- Another famous chaotic system:

- Rössler system:  $\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$

only nonlinearity

(e.g., it has a strange attractor for  
 $a = b = 0.2, c = 5.7$ )

- 'Simple' conservative chaotic systems include forced pendulum (recall last 2), double pendulum, & Hénon-Heiles system (I'll present the last of these here)
- Consider a ring of 3 particles connected by "nonlinear springs" (springs w/ nonlinear restoring forces)

it has Hamiltonian  $H = \frac{1}{2} \sum p_k^2 + \frac{1}{2} \sum (Q_{k+1} - Q_k)^2 + \frac{\omega}{3} \sum (Q_{k+1} - Q_k)^3$  (75)

$\tau = 3$ -particle FPU system <sup>Fermi-Pasta-Ulam</sup> all sums go from  $k=1$  to  $k=3$ ,  
w/ 'periodic boundary conditions' w/  $Q_4 = Q_1$ , 

- a change of var. (to "harmonic normal mode coordinates") gives

$$H = \frac{1}{2} (\tilde{P}_1^2 + \tilde{P}_2^2 + \tilde{P}_3^2) + \frac{1}{2} (3\tilde{Q}_1^2 + 3\tilde{Q}_2^2) + \left( \frac{3\omega}{\sqrt{2}} \right) (\tilde{Q}_1 \tilde{Q}_2^2 - \underbrace{\frac{1}{3} \tilde{Q}_3^2}_{\text{old var}}) \quad (76)$$

- $H$  does not depend on  $\tilde{Q}_3$ , which locates the center of mass, so the center of mass move is const. momentum  $\tilde{P}_3$  (in a Hamiltonian system, if a var is absent from the Hamiltonian, then its conjugate variable must be a const.  $\rightarrow$  follows from Hamilton's eqns.; check this at home)

53a

 MAP  
 2/2/10  
 B2b

→ transform to a frame moving w/ the center of mass & set  $t = \frac{z}{\sqrt{3}}$ ,  $Q_2 = \frac{\sqrt{2}}{2} \dot{q}_2$ ,  
 $\tilde{Q}_3 = \frac{\sqrt{2}}{2} \dot{q}_3$ ,

$$\therefore H = \frac{1}{2} (p_1^2 + p_2^2 + \dot{q}_1^2 + \dot{q}_2^2) + \dot{q}_1 \dot{q}_2 - \frac{1}{3} \dot{q}_3^2 \quad (77)$$

↑ Hénon-Heiles Hamiltonian, whose chaotic properties have been exhaustively investigated

(recall: trajectories in a Hamiltonian system have  $H = \text{const.}$ )

- (76) has 2 degrees-of-freedom (DOF)
- $(q_1, p_1)$  is one pair of vars (aka, 1 DOF) and  $(q_2, p_2)$  is the other
- H & H studies the global behavior of orbits in (77) using Poincaré sections
- $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$  but because  $H = E = \text{energy}$  is a const. of motion, we can draw orbits in  $(\dot{q}_1, \dot{q}_2, p_2)$ -space because w/ the coordinates given, we can subsequently determine  $p_1 > 0$  (or  $p_1 < 0$ ) using  $E = E(q_1, q_2, p_1, p_2)$
- ↳ but the  $(\dot{q}_2, p_2)$  plane provides a non-elliptic (a Poincaré section) of the  $(q_1, q_2, p_2)$ -space
- ∴ can obtain a global picture of the dynamics at each energy by determining each orbit's intersection per. w/ the  $(\dot{q}_2, p_2)$ -plane — i.e.,  $(q_2, p_2)$ -coordinates for which  $\dot{q}_1 = 0$  and  $p_1 > 0$  (need to choose circle  $p_1 < 0$  or  $p_1 > 0$ ; this indicates direction we pass through  $\dot{q}_1 = 0$ )

536

- if (77) is chaotic w/ no const of motion besides  $E$ , then the Poincaré section would give a splatter of dots w/ no apparent pattern
- if (77) is integrable in the sense of having the same number of const. of motion as DOF (i.e., a constant in addition to  $E$ ),  
 $\Phi = \Phi(E, \varepsilon_1, \varepsilon_2, p_1, p_2)$   
then by find  $E$  for  $p_1 > 0$  & plugging in, we'd find  
 $\Phi = \Phi(E, \varepsilon_1, \varepsilon_2, p_2)$ , an eqn for a 2D surface embedded in  $(\varepsilon_2, p_2)$ -space (which is  $\mathbb{R}^3$ )  

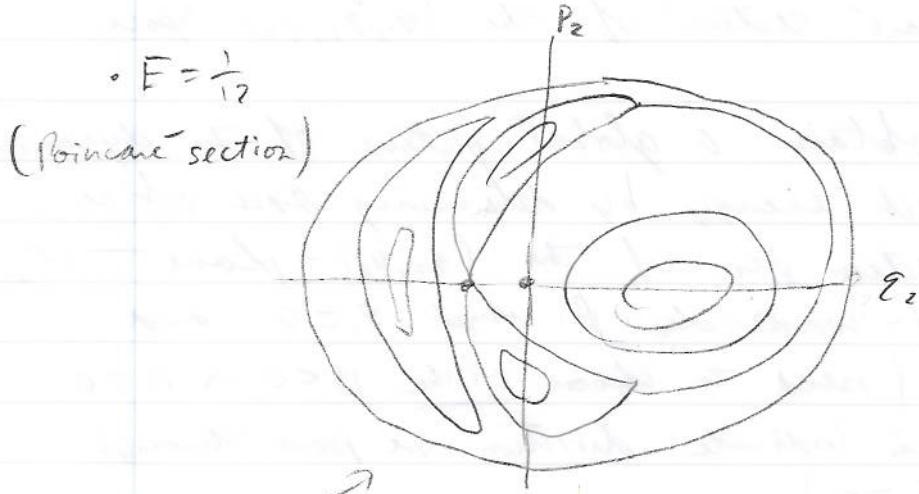
↓

setting  $\varepsilon_1 = 0$  in  $\Phi$ , we can determine the curve  
 $\Phi = \Phi(E, \varepsilon_2, p_2)$  in the  $(\varepsilon_2, p_2)$ -plane on which the intersection of trajectories must lie

↓

∴ no matter whether integrable or chaotic, Hénon-Heiles only had to numerically compute the trajectories of (77), determining  $(\varepsilon_2, p_2)$ -intersection pts, in order to characterize the system motion at each energy value

- as energy gets bigger, the contribution of nonlinear terms increasingly perturb the (harmonic) quadratic terms, so HH used energy as a perturbation param



(supposed to be symmetric about  $\varepsilon_2$ -axis, if I had drawn it well) (which is quasiperiodic)

- curves appear to exist everywhere, indicating possible integrability

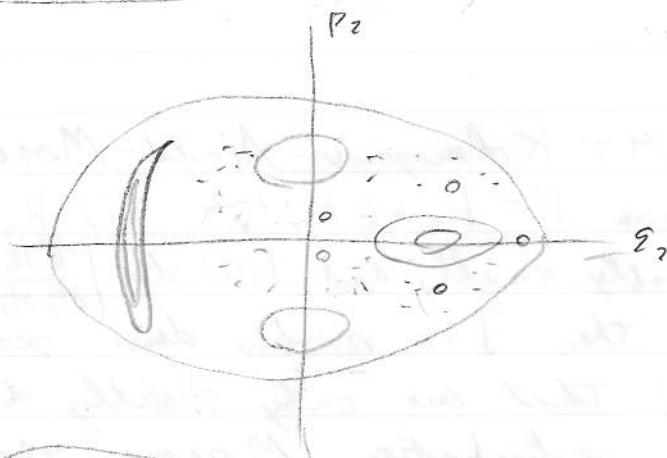
- each curve comes from a single trajectory

54a

MP0  
2/5/10  
B.8b

- Let's increase the energy:

$$E = \frac{1}{8}$$



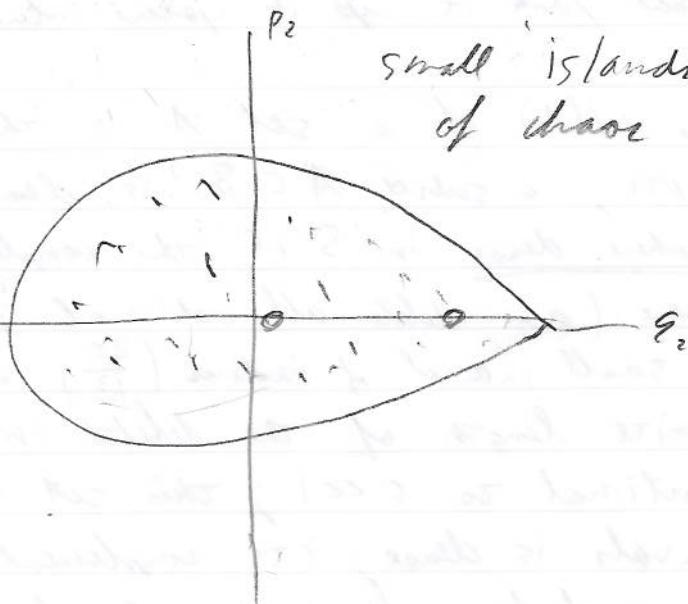
but note that region of chaotic & regular behavior coexist in a complicated fashion!

$$f_n E = \frac{1}{8},$$

• curves persist in the neighborhood of the stable equilibria along the  $q_2$  &  $p_2$  axes (these stable equilibria — at the centers of the curves in the Poincaré section — are centers because the system is Hamiltonian & have nearby periodic orbits), but one can see erratic dots (chaotic area) near these stable regions & we have sensitive dependence of nearby trajectories in this chaotic region

- now increase the energy again:

$$E = \frac{1}{6}$$



small 'islands' in a sea of chaos

(these islands are 'riri' because of the quasiperiodic nature of such curves, so the trajectories live on a torus)

59b) This type of perturbation to a 2 DOF Hamiltonian system can be formalized & has led to much theoretical work:

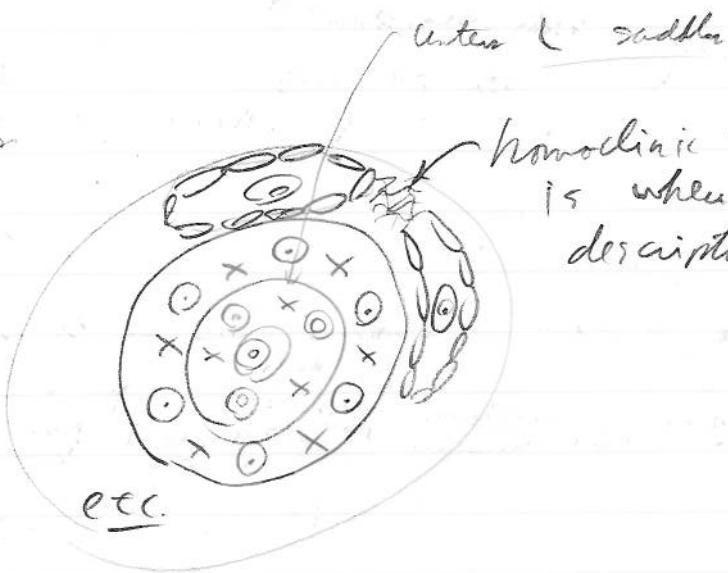
"KAM Theorem" [KAM = Kolmogorov-Arnold-Moser]: Given an analytic Hamiltonian  $H = H_0 + \epsilon H_1$ , s.t. (i)  $H_0$  is integrable, (ii)  $\epsilon$  is sufficiently small, and (iii)  $\det \left| \frac{\partial^2 H_0}{\partial P_k \partial P_l} \right| = \det \left| \frac{\partial \omega_k}{\partial P_l} \right| \neq 0$  (where  $\omega_k = \frac{\partial H_0}{\partial P_k}$ ), then there is a nowhere dense set of  $H_0$ -tori (called "KAM tori") that are only slightly distorted by the small  $\epsilon H_1$  perturbation. Moreover, the measure of this nowhere dense set of "preserved" tori is nearly that of the allowed phase space. The complementary set of "destroyed"  $H_0$ -tori is dense but has small measure.

Notes:

- (1) This theorem can be stated more mathematically. :)
- (2)  $H_0$  is integrable means if  $H_0$  has n DOF, then const. of motion
- (3) condition 3 is a 'twist condition' ensuring nondegeneracy so that the perturbative argument can hold  
(I haven't given enough details for you to understand this, but look it up if you're interested)
- (4) the closure  $\bar{A}(S)$  of a set  $A$  is the union of  $A$  & its limit pts.; a subset  $A \subset S$  is dense<sup>in S</sup> if  $\bar{A} = S$ ;  $A$  is nowhere dense in  $S$  if the complement of  $\bar{A}$  is dense there (e.g., delete all rationals  $\frac{p}{k}$  from  $[0,1]$  & also delete a small interval of radius  $(\frac{2\epsilon}{k^3})$  about each rational; the finite length of the deleted intervals is proportional to  $\epsilon^{kk!}$ ; this set of deleted intervals is dense; its complement is nowhere dense yet includes almost the entire length of  $[0,1]$ )

(5) for continuous pt. sets, measure simply mean length, area, volume, etc. For discontinuous or disjoint point sets, we can cover such sets w/ a collection of length, area, volume, etc. and we can then determine volume, etc. of the limiting collection of elements that just cover the given set — we then call this the measure of the set (e.g., the rationals  $\frac{p}{q}$  in the above example can be covered by intervals of radius  $\frac{2\epsilon}{q^3}$  whose total length we can make as small as we want; hence the set of rational numbers has measure 0)

(cont'd)  
 Prime section  
 of mixture  
 of regular &  
 chaotic  
 dynamics



- Now we'll move on to some other cool topics in nonlinear systems ...

