B8b Nonlinear Systems (Mock Exam)

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Do not turn this page until you are told that you may do so

1. (i) Define a *saddle-node bifurcation* and show that the first order system

$$\frac{dx}{dt} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation as the real parameter r is varied.

(ii) Consider the linear dynamical system

$$\begin{aligned} \frac{dx}{dt} &= ax + by\,, \\ \frac{dy}{dt} &= cx + dy\,, \end{aligned} \tag{1}$$

where a, b, c, and d are real constants. Show that the eigenvalues of the associated Jacobian matrix are completely determined by the trace and determinant of the Jacobian matrix. Use this result to classify the qualitative behavior of equilibria of (1) in the trace-determinant plane. Comment on when this behavior is representative of a nonlinear system whose linearization is given by (1).

(iii) Consider the system of phase oscillators

$$\begin{aligned} \frac{d\theta_1}{dt} &= \omega_1 + \alpha_1 \sin(\theta_2 - \theta_1)\,, \\ \frac{d\theta_2}{dt} &= \omega_2 + \alpha_2 \sin(\theta_1 - \theta_2)\,. \end{aligned} \tag{2}$$

Derive a condition for the motion to be phase locked and also derive a formula for the common locked frequency.

(iv) Suppose there are three phase-only oscillators θ_1, θ_2, and θ_3 that have respective natural frequencies ω_1, ω_2, and ω_3. Suppose further that when uncoupled, the three equations of motion each have the form $\frac{d\theta_i}{dt} = \omega_i$. If one oscillator is coupled to another only via terms like $K \sin(\theta_i - \theta_j)$ (for $i \neq j$), write down the equations of motion for (a) the three oscillators coupled together in a line and (b) the three oscillators coupled together in a ring. Also give a physical interpretation for the coupling term $K \sin(\theta_i - \theta_j)$. In the case in which $\omega_1 = \omega_3$ and the oscillators are coupled in a line, show that there is a stable phase-locked solution when $\rho := (\omega_2 - \omega_1)/K < 3$. *[You may assume that* $\rho \geqslant 0$.*]*

2. (i) Consider the nonlinear two-dimensional autonomous dynamical system

$$\frac{dx}{dt} = f(x, y)\,,$$
$$\frac{dy}{dt} = g(x, y)\,, \tag{3}$$

where t denotes time, and suppose that it has an equilibrium at the point (x_0, y_0). Linearize (3) about (x_0, y_0) to *derive* the usual linear stability conditions. Use your derivation to explain what happens when at least one eigenvalue of the Jacobian has a zero real part. What, if anything, changes if (3) is n-dimensional (for arbitrary n) instead of two-dimensional? What, if anything, changes if the right-hand-side of (3) has a term with an explicit time-dependence?

(ii) State the Poincaré-Bendixson theorem.

(iii) Use multiple timescale perturbation theory (with fast time $\tau = t$ and slow time $T = \epsilon t$) to show that the van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{4}$$

has for $0 < \epsilon \ll 1$ a stable limit cycle that is nearly circular, with radius of $2 + O(\epsilon)$ and a frequency of $\omega = 1 + O(\epsilon^2)$. [Recall that the expansion is $x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$.]

3. (i) Consider the map $x_{n+1} = f(x_n) = \lambda e^{x_n}$ for $\lambda > 0$. Classify the bifurcation at $\lambda = 1/e$.

(ii) Suppose you are given the map $x_{n+1} = f(x_n) = r - x_n^2$ and the equation

$$f(x, R_\infty) \approx \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right)\,, \tag{5}$$

where α is a constant and $r = R_\infty$ designates the lowest value of r at which chaotic dynamics first occurs. State briefly what equation (5) means. Also state the meaning and illustrate with plots the meaning of the equation

$$f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right)\,, \tag{6}$$

where $r = R_n$ (for finite n) denotes the lowest value of r at which a 2^n-cycle becomes superstable.

(iii) Use symbolic dynamics to show that the binary shift map $y_{n+1} = f(y_n) = 2y_n$ (mod 1) has (a) a countably infinite number of periodic orbits, (b) an uncountable number of nonperiodic orbits, (c) sensitive dependence on initial conditions, and (d) a dense orbit. [Note: These conditions together guarantee that the map is chaotic.]