Solutions for B8b (Nonlinear Systems) "Fake Past Exam" (TT 10)

Mason A. Porter

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1 Question 1

i. (6 points) Define a *saddle-node bifurcation* and show that the first order system

$$\frac{dx}{dt} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation as the real parameter r is varied.

Solution: The saddle-node bifurcation is the simplest mechanism by which equilibria are created or destroyed. There is a single bifurcation parameter (i.e., it has codimension 1); call it r. On one side of the bifurcation value, there are 2 equilibria (a saddle and a node—so one unstable one and one stable one) and on the other side, there are none.

In the equation of interest, equilibria satisfy $f(x) = r - x - e^{-x} = 0$. We can solve this equation in closed form, but we can find the points geometrically: plot $g_1(x) = r - x$ and $g_2(x) = e^{-x}$ on the same coordinates and see where they intersect. Those intersections give the desired equilibria. This also allows us to tell stability because the flow is to the right when the line is above the curve (as $\dot{x} > 0$ in this case) and to the left when it is below it. As we decrease the parameter r, the line $r - x$ slides down, so we go from 2 intersections to 1 (tangent) intersection to 0 intersections. As we can see in Fig. 1, a saddle and a node coalesce in this bifurcation.

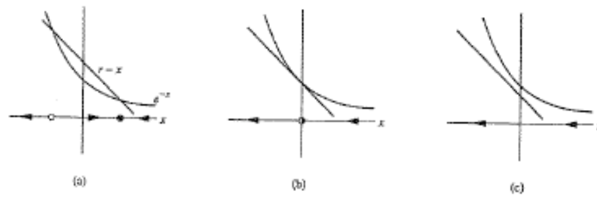

Figure 1: Saddle-node bifurcation for part (i).

ii. (6 points) Consider the linear dynamical system

$$\frac{dx}{dt} = ax + by\,,$$
$$\frac{dy}{dt} = cx + dy\,, \tag{1}$$

where a, b, c, and d are real constants. Show that the eigenvalues of the associated Jacobian matrix are completely determined by the trace and determinant of the Jacobian matrix. Use this result to classify the qualitative behavior of equilibria of (1) in the trace-determinant plane. Comment on when this behavior is representative of a nonlinear system whose linearization is given by (1).

Solution: The Jacobian matrix is

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{2}$$

Calculating the eigenvalues gives

$$\lambda^2 - \tau\lambda + \Delta = 0\,, \tag{3}$$

where

$$\tau = \text{trace}(J) = a + d\,,$$
$$\Delta = \det(J) = ad - bc\,. \tag{4}$$

Therefore, the eigenvalues are

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}\,. \tag{5}$$

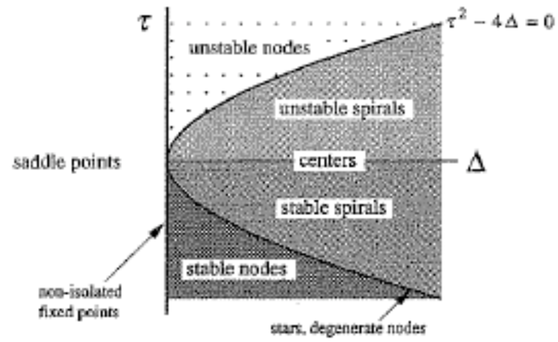

Figure 2: Classification of equilibria for part (ii).

One thereby obtains the classification show in Fig. 2 It's useful to note that $\Delta = \lambda_+\lambda_-$, which one can obtain from (3).

If $\Delta < 0$, eigenvalues are real and have opposite sign, which gives a saddle.

If $\Delta > 0$, eigenvalues are either real with the same sign (nodes) or complex conjugate (spirals or centers). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ contains star nodes and degenerate nodes. (I don't think I gave you the term "star", so I'd only expect a brief mention of degeneracy.) The stability of nodes and spirals is determined by the sign of τ. When $\tau < 0$, the equilibrium is stable; when $\tau > 0$, it is unstable.

If $\Delta = 0$, at least one eigenvalue is 0. One then gets either a line of equilibria or an entire plane of them (if $J = 0$).

For a nonlinear system, such a classification arises from calculation of *linear stability*, which is a *local stability*. The nonlinear system is a small perturbation of the linear one as long as one is sufficiently near an equilibrium. If no eigenvalues have 0 real part, then the stability results carry through sufficiently near the equilibrium, as the perturbation is not large enough to change the sign of the real part. Any equilibrium with eigenvalue with 0 real part could might be either stable (possibly asymptotically stable or maybe just neutrally stable) or unstable based on the results of the linearization, so one has to consider the nonlinearity to determine the stability.

iii. (6 points) Consider the system of phase oscillators

$$\frac{d\theta_1}{dt} = \omega_1 + \alpha_1 \sin(\theta_2 - \theta_1)\,,$$
$$\frac{d\theta_2}{dt} = \omega_2 + \alpha_2 \sin(\theta_1 - \theta_2)\,. \tag{6}$$

Derive a condition for the motion to be phase locked and also derive a formula for the common locked frequency.

Solution: To study (6), define the phase lag $\phi := \theta_1 - \theta_2$. This yields

$$\frac{d\phi}{dt} = \omega_1 - \omega_2 - (\alpha_1 + \alpha_2)\sin\phi\,, \tag{7}$$

which is defined on the circle (i.e., $\phi \in S^1$). An equilibrium of (7) corresponds to a phase-locked solution of (6). Setting $\dot{\phi} = 0$ gives

$$\sin\phi = \frac{\omega_1 - \omega_2}{\alpha_1 + \alpha_2}\,. \tag{8}$$

We need these roots to be real, so we obtain the following condition to obtain phase-locked motion:

$$\left|\frac{\omega_1 - \omega_2}{\alpha_1 + \alpha_2}\right| \leq 1\,. \tag{9}$$

That is, phase locking requires the difference in uncoupled frequencies to be small relative to the sum of the coupling constants. Substituting the expression (8) into (6) gives the common locked frequency, which is the weighted average of the uncoupled frequencies:

$$\dot{\theta}_1 = \dot{\theta}_2 = \frac{\alpha_1\omega_2 + \alpha_2\omega_1}{\alpha_1 + \alpha_2}\,. \tag{10}$$

iv. (7 points) Suppose there are three phase-only oscillators θ_1, θ_2, and θ_3 that have respective natural frequencies ω_1, ω_2, and ω_3. Suppose further that when uncoupled, the three equations of motion each have the form $\frac{d\theta_i}{dt} = \omega_i$. If one oscillator is coupled to another only via terms like $K\sin(\theta_i - \theta_j)$ (for $i \neq j$), write down the equations of motion for (a) the three oscillators coupled together in a line and (b) the three oscillators coupled together in a ring. Also give a physical interpretation for the coupling term $K\sin(\theta_i - \theta_j)$. In the case in which $\omega_1 = \omega_3$ and the oscillators are coupled in a line, show that there is a stable phase-locked solution when $\rho := (\omega_2 - \omega_1)/K < 3$. *[You may assume that $\rho \geq 0$.]*

Solution: This comes from problem 3a,b from problem sheet 2.

The equations of motion for three oscillators coupled in a line are

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + K\sin(\theta_2 - \theta_1)\,, \\
\dot{\theta}_2 &= \omega_2 + K\sin(\theta_1 - \theta_2) + K\sin(\theta_3 - \theta_2)\,, \\
\dot{\theta}_3 &= \omega_3 + K\sin(\theta_2 - \theta_3)\,.
\end{aligned} \tag{11}$$

Note that only the center oscillator (which I have labeled "2") is coupled to both of the other oscillators. In a ring, on the other hand, each of the 3 oscillators is coupled to the other two oscillators:

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + K\sin(\theta_2 - \theta_1) + K\sin(\theta_3 - \theta_1)\,, \\
\dot{\theta}_2 &= \omega_2 + K\sin(\theta_1 - \theta_2) + K\sin(\theta_3 - \theta_2)\,, \\
\dot{\theta}_3 &= \omega_3 + K\sin(\theta_2 - \theta_3) + K\sin(\theta_1 - \theta_3)\,.
\end{aligned} \tag{12}$$

The coupling term $K\sin(\theta_i - \theta_j)$ means that oscillator j only cares about its phase difference with oscillator i. The prefactor K indicates how strong this coupling is, and the strength of the coupling gets larger (and more positive) when i is farther ahead of j (until it gets more than π ahead) and gets larger (and more negative) when i is farther behind j (again, until the difference is more than π).

The remaining text comes from the problem sheet solutions (with some adaptation): The relative phases are $\phi_1 := \theta_2 - \theta_1$ and $\phi_2 := \theta_2 - \theta_3$. We rescale time with $t \to t'K$ and introduce the parameter $\rho := (\omega_2 - \omega_1)/K = (\omega_2 - \omega_3)/K$. This yields the system

$$\begin{aligned}
\dot{\phi}_1 &= \rho - 2\sin\phi_1 - \sin\phi_2\,, \\
\dot{\phi}_2 &= \rho - 2\sin\phi_2 - \sin\phi_1\,,
\end{aligned} \tag{13}$$

where we have for convenience renamed t' as t. The equilibria of (13), which you should note is defined on the torus (because both variables are angular), give conditions for phase locking in (11), as this implies that the phases have a constant difference and travel at the same frequency.

Equilibria satisfy $\dot{\phi}_1 = \dot{\phi}_2 = 0$, and they exist provided $|\rho| \leq 3$. When $|\rho| < 3$, there are four equilibria, each of which satisfies $3\sin\phi^* = \rho$.

When $|\rho| = 3$, there is a single equilibrium at $\text{sign}(\rho)(\pi/2, \pi/2)$, as a bifurcation occurs for that value of ρ. We can in fact consider $\rho \geq 0$ without loss of generality because we can just apply the reflection $\phi_1' = -\phi_1$ and $\phi_2' = -\phi_2$, and obtain a system with the same properties as the original system. (In the question, I purposely specified that you need not consider negative ρ.)

2 Question 2

i. (8 points) Consider the nonlinear two-dimensional autonomous dynamical system

$$\frac{dx}{dt} = f(x, y)\,,$$
$$\frac{dy}{dt} = g(x, y)\,, \tag{14}$$

where t denotes time, and suppose that it has an equilibrium at the point (x_0, y_0). Linearize (14) about (x_0, y_0) to *derive* the usual linear stability conditions. Use your derivation to explain what happens when at least one eigenvalue of the Jacobian has a zero real part. What, if anything, changes if (14) is n-dimensional (for arbitrary n) instead of two-dimensional? What, if anything, changes if the right-hand-side of (14) has a term with an explicit time-dependence?

Solution: Because (x_0, y_0) is an equilibrium point, we have $f(x_0, y_0) = g(x_0, y_0) = 0$. Let $u = x - x_0$ and $v = y - y_0$ denote the components of a small disturbance from the equilibrium point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. The equation for u is

$$\begin{aligned} \dot{u} &= \dot{x} \\ &= f(x_0 + u, y_0 + v) \\ &= f(x_0, y_0) + uf_x + vf_y + O(u^2, v^2, uv)\,. \end{aligned} \tag{15}$$

In the Taylor expansion above, note that f_x and f_y are evaluated at (x_0, y_0). Similarly, we get

$$\dot{v} = g(x_0, y_0) + ug_x + vg_y + O(u^2, v^2, uv)\,. \tag{16}$$

We thus get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms}\,. \tag{17}$$

The matrix of partial derivatives in (17) is the Jacobian matrix J. This, then, gives the usual stability conditions in terms of the eigenvalues of J (evaluated at the equilibrium in question). When those eigenvalues all have negative real part, the equilibrium is (locally) asymptotically stable. When at least one has a positive real part, it's unstable. An eigenvalue that has zero real part is marginally stable. It is stable in the linearization, but perturbations can either cause it to become positive, become negative, or remain zero, so one has to deal with the nonlinearity to actually determine stability.

Nothing changes if the system is n-dimensional for integers $n > 2$. (If $n = 1$, there is just a single eigenvalue, which is J itself.) Things like stable, unstable, and center manifolds can of course become more complicated—and more complicated types of bifurcations can occur—but looking at the real parts of eigenvalues remains the same.

If there is an explicit time-dependence in the right-hand-side, one can't apply these considerations, although some generalizations (not discussed in the course) do exist.

Note: This is a question that could have included my asking a qualitative question about things like bifurcations or stable/unstable manifolds.

ii. (8 points) State the Poincaré-Bendixson theorem.

Solution: This answer is directly out of lectures. To get the full 5 points, the statement has to be pretty pristine (though it does *not* need to be word-for-word from the lectures), but partial credit would be awarded for partially correct statements—e.g., ones that get certain salient features correct (the fact that this theorem is specifically for planar regions) but not others. Anyway, on to the theorem statement...

Suppose that:

1. R is a closed, bounded subset of the plane;
2. $\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R; note importantly that $x \in \mathbb{R}^2$ and $f \in \mathbb{R}^{\nvDash}$;

3. R does not contain any equilibrium points; and
4. there exists a trajectory C that is "confined" in R in the sense that it starts in R and stays in R for all future time.

Then either C is a closed orbit or it spirals towards a closed orbit as $t \to \infty$. (In either case R *contains a closed orbit.*)

Note: This theorem inherently requires the topology of $\mathbb{R}^2$ to work. I could have asked something related to this in the context of, say, chaotic orbits in vector fields in $\mathbb{R}^3$.

iii. (9 points) Use multiple timescale perturbation theory (with fast time $\tau = t$ and slow time $T = \epsilon t$) to show that the van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{18}$$

has for $0 < \epsilon \ll 1$ a stable limit cycle that is nearly circular, with radius of $2 + O(\epsilon)$ and a frequency of $\omega = 1 + O(\epsilon^2)$. [Recall that the expansion is $x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$.]

Solution: First, we need to use the chain rule:

$$\dot{x} = \partial_\tau x + \epsilon \partial_T x\,, \tag{19}$$

and similarly for $\ddot{x}$. Note the shorthand that I am using for the partial derivatives. Using the given expansion, we have

$$\begin{aligned} \dot{x} &= \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2)\,, \\ \ddot{x} &= \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\epsilon^2)\,. \end{aligned} \tag{20}$$

We then need to collect terms in like powers of ϵ order by order. At $O(1)$, we get

$$\partial_{\tau\tau} x_0 + x_0 = 0\,, \tag{21}$$

and at $O(\epsilon)$ we get

$$\partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - (x_0^2 - 1)\partial_\tau x_0\,. \tag{22}$$

The $O(1)$ is of course the simple harmonic oscillator, so it has solution

$$x_0 = r(T)\cos(\tau + \phi(T))\,, \tag{23}$$

where $r(T)$ is a slowly varying amplitude and $\phi(T)$ is a slowly varying phase.

To find equations governing r and ϕ, we insert (23) into (22). This yields

$$\partial_{\tau\tau}x_1 + x_1 = -2(r'\sin(\tau+\phi) + r\phi'\cos(\tau+\phi)) - r\sin(\tau+\phi)[r^2\cos^2(\tau+\phi) - 1]\,, \tag{24}$$

where $' = d/dT$. We need to avoid resonant terms on the right-hand-side; these are the terms proportional to $\sin(\tau+\phi)$ and $\cos(\tau+\phi)$. Note that we need to do a Fourier expansion on the right-hand-side (which, in practice, means just using a trig identity in this case). One can either remember or derive the identity (say, by justing do the computation with complex exponentials)

$$\sin(\tau+\phi)\cos^2(\tau+\phi) = \frac{1}{4}[\sin(\tau+\phi) + \sin(3[\tau+\phi])]\,. \tag{25}$$

We thus have

$$\partial_{\tau\tau}x_1 + x_1 = [-2r' + r - \frac{1}{4}r^3]\sin(\tau+\phi) + [-2r\phi']\cos(\tau+\phi) - \frac{1}{4}r^3\sin[3(\tau+\phi)]\,. \tag{26}$$

To avoid secular (aka, resonant) terms, we thus require that

$$\begin{aligned} -2r' + r - \frac{1}{4}r^3 &= 0\,, \\ -2r\phi' &= 0\,. \end{aligned} \tag{27}$$

The first equation in (27), which is defined on the half line, can be written

$$r' = \frac{r}{8}(4 - r^2)\,. \tag{28}$$

We find equilibria with $r' = 0$ and see that $r = 0$ is unstable but $r = 2$ is stable. Hence, $r(T) \to 2$ as $T \to \infty$. The second equation in (27) implies that $\phi' = 0$ (unless $r = 0$), so $\phi(T) = \phi_0 =$ constant. Hence, $x_0(\tau, T) \to 2\cos(\tau + \phi_0)$ and thus

$$x(t) \to 2\cos(t + \phi_0) + O(\epsilon) \tag{29}$$

as $t \to \infty$. Thus, $x(t)$ approaches a stable limit cycle of radius $2 + O(\epsilon)$.

To find the frequency of the limit cycle, let $\theta = t + \phi(T)$ denote the argument of the cosine. The angular frequency is then

$$\omega = \dot{\theta} = 1 + \phi'\dot{T} = 1 + \epsilon\phi' = 1 + O(\epsilon^2)\,. \tag{30}$$

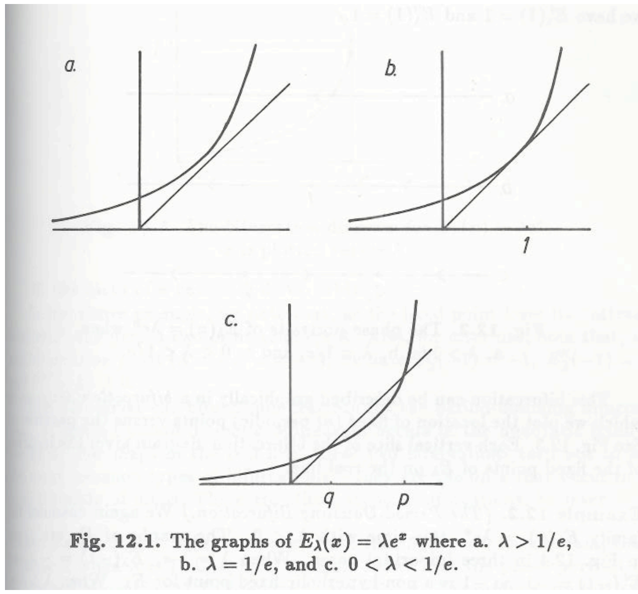

Figure 3: Graphs of $f(x) = \lambda e^x$ for $\lambda > 0$.

3 Question 3

i. (10 points) Consider the map $x_{n+1} = f(x_n) = \lambda e^{x_n}$ for $\lambda > 0$. Classify the bifurcation at $\lambda = 1/e$.

Solution: This was problem 1a on homework sheet 3. Below I repeat the text and graphics from that answer sheet.

As depicted in Fig. 3, this family of functions has a bifurcation at $\lambda = 1/e$. (One looks at the relation between the graph of $f(x)$ and the graph of $g(x) = x$.)

When $\lambda > 1/e$, the graph of f does not intersect the diagonal, so f has no fixed points. When $\lambda = 1/e$, the graph of f meets the diagonal $g(x) = x$ tangentially at $(x, y) = (1, 1)$. When $\lambda < 1/e$, the graph intersects the diagonal at 2 points—at q such that $f'(q) < 1$ and at p such that $f'(p) > 1$. Hence, f has two fixed points when $\lambda < 1/e$. By looking at the graphs (and using, e.g., cobwebbing), one obtains the following observations:

1. When $\lambda > 1/e$, $f^n(x) \to \infty$ for all x.

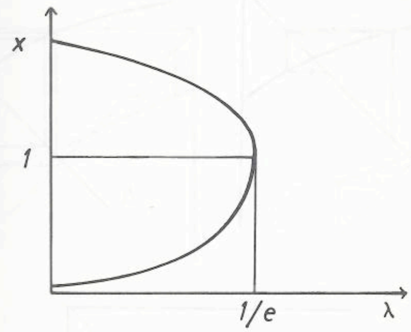

Fig. 12.3. **The bifurcation diagram for $E_\lambda(x) = \lambda e^x$; x is plotted versus λ.**

Figure 4: Bifurcation diagram for the map $f(x) = \lambda e^x$ for $\lambda > 0$. Note that the vertical axis gives the location of the fixed points.

2. When $\lambda = 1/e$, then $f(1) = 1$. If $x < 1$, then $f^n(x) \to 1$. If $x > 1$, then $f^n(x) \to \infty$.
3. When $0 < \lambda < 1/e$, then $f(q) = q$ and $f(p) = p$. If $x < p$, then $f^n(x) \to q$. If $x > p$, then $f^n(x) \to \infty$.
4. At the bifurcation, we have $f'(1) = 1$ and $f''(1) = 1$.

What we have seen in this problem is a *saddle-node (or "tangent") bifurcation of maps.* We plot the bifurcation diagram in Fig. 4.

ii. (5 points) Suppose you are given the map $x_{n+1} = f(x_n) = r - x_n^2$ and the equation

$$f(x, R_\infty) \approx \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right) , \tag{31}$$

where α is a constant and $r = R_\infty$ designates the lowest value of r at which chaotic dynamics first occurs. State briefly what equation (31) means. Also state the meaning and illustrate with plots the meaning of the equation

$$f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right) , \tag{32}$$

where $r = R_n$ (for finite n) denotes the lowest value of r at which a 2^n-cycle becomes superstable.

Solution: This was discussed in the lecture notes.

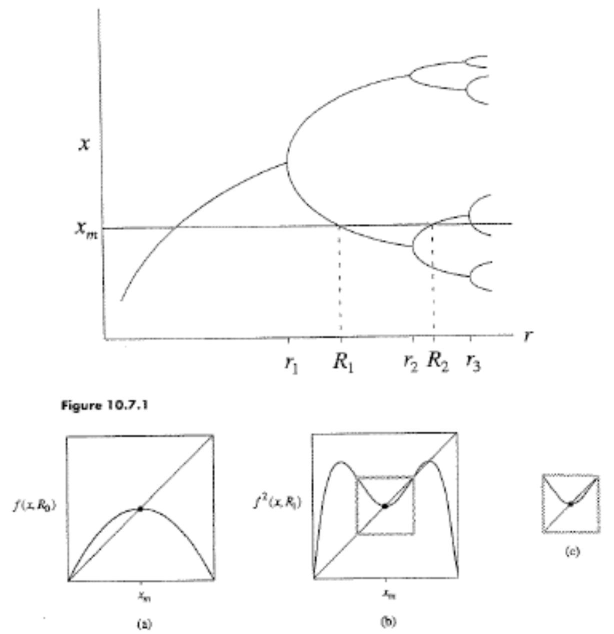

Figure 5: Bifurcation diagram and iterates of the map f.

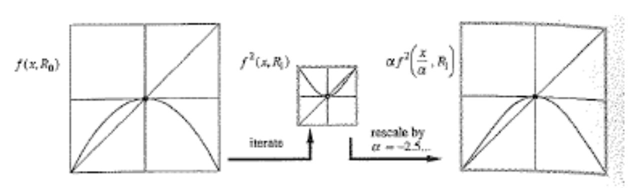

Figure 6: Zooming in to illustrate renormalization.

Let's answer the second part of the equation first. Take a look at Figs. 5, 6. Iterating the quadratic map f twice (i.e., applying the map f^2) gives a quartic map. If we zoom in on the local minimum by some amount and take a negative sign so that it's a local maximum instead of a local minimum (these two operations together determine α) gives a window that looks similar to the original window. That is the equality in (32). Additionally, we need to advance which point we're talking about (one with twice the periodicity) because the map is f^2 rather than f. At superstable fixed points, the Jacobian (i.e., the eigenvalue) is 0. A superstable cycle of a unimodal map like f always contains x_m as one of its points, as depicted in Fig. 5, so in particular x_m is a superstable point in a cycle for *both* f and f^2, and this allows us to compare them as in (32). Note that $r = r_n$ denotes the smallest value of r at which a 2^n cycle is born. That is where the branching in the diagram occurs. Recall that α is universal for generic unimodal maps (generic in the sense that the extremum has a nonzero quadratic term in its Taylor series).

Equation (31) is the limit of equations like (32) that is valid at the onset of chaos. The nice thing here is that R_∞ is on both sides of the equation, so r no longer needs to be shifted when we renormalize. (Remember that consecutive R_n get progressively closer to each other as one iterates.) This then gives us a functional differential equation for a universal function $g(x)$. (See the notes for how it's defined.) One can study this equation—with additional boundary conditions that are discussed in the notes—to try to estimate the value of the universal scale factor α and also the value of the universal Feigenbaum constant δ.

Note: This answer has more detail than I'd expect, but I would expect appropriate diagrams and comments about what things like α, etc. are doing.

iii. (10 points) Use symbolic dynamics to show that the binary shift map $y_{n+1} = f(y_n) = 2y_n \pmod 1$ has (a) a countably infinite number of periodic orbits, (b) an uncountable number of nonperiodic orbits, (c) sensitive dependence on initial conditions, and (d) a dense orbit. [Note: These conditions together guarantee that the map is chaotic.]

Solution: Note that we can write y_n as a binary expansion

$$y_n = \sum_{k=1}^{\infty} \frac{a_k}{2^k}\,, \tag{33}$$

where $a_k \in \{0, 1\}$.

a. A point y is a period-n cycle of f if

$$f^n(x) = 2^n x \bmod 1\,. \tag{34}$$

We thus need $0 = f^n(x) - x = 2^n x - x - k$ (for some integer k), which implies that

$$x = \frac{k}{2^n - 1}\,. \tag{35}$$

That is, we need x to be rational.

As x is rational, we write it as $x = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. We consider $2^m p \pmod q$ for nonnegative integer m. When considering the quantity $2^m p$, we note that we can only obtain at most q possibilities after taking things mod q. Hence, for some nonnegative integers m and n, we have

$$\begin{aligned}
&2^m p = 2^n p (\bmod q) \\
&\implies 2^m (p/q) = 2^n (p/q) (\bmod 1) \\
&\implies 2^m x = 2^n x (\bmod 1) \\
&\implies (2^m - 2^n) x = k \in \mathbb{Z} \\
&\implies x = \frac{k 2^{-n}}{2^{m-n} - 1}\,,
\end{aligned} \tag{36}$$

so that the symbol sequence of $f^n y_n$ repeats after a finite number of iterations (i.e., the orbit is periodic). As $n \in \mathbb{N}$, we have a countably infinite number of periodic orbits.

b. Irrational numbers are an uncountable set and they are given by the symbol sequences with non-terminating binary expansions:

$$y_n = .a_1 a_2 \cdots\,. \tag{37}$$

There are uncountably infinitely many points with such expansions, so there are uncountably infinitely many irrational orbits.

c. Consider two nearby points

$$y_1 = .a_1 \cdots a_n c\,,$$
$$y_2 = .a_1 \cdots a_n d\,, \tag{38}$$

with $c \neq d$. After n iterations, we have $f^n(y_1) = c$ and $f^n(y_2)$, so now the two points have eventually become very far away from each other. By making n larger, we see that this occurs for points that start out arbitrarily close to each other, which gives sensitive dependence on initial conditions.

d. Consider a concatenation of all periodic orbits:

$$y = .0, 1, 00, 01, 10, 11, \cdots\,. \tag{39}$$

This point y visits a neighborhood of every periodic orbit, so this orbit is dense (as it spends time arbitrarily close to any given sequence). [Recall that a subset is dense if its closure is the entire space.]