# Synchronization in Coupled Phase Oscillators

Natasha Cayco Gajic

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**Abstract**

In a system of coupled oscillators, *synchronization* occurs when the oscillators spontaneously lock to a common frequency or phase. We study a system of $n \gg 1$ phase oscillators placed on a circle with random initial positions and sinusoidal coupling with their $k$ nearest neighbors on each side. When all of the oscillators are identical, the final state of the system is either full phase-locking (in which all oscillators have the same relative phase) or a splay state characterized by a winding number $q$ with the oscillators uniformly spread apart in phase. However, when the internal frequencies of the oscillators are uniformly distributed on a small interval, we demonstrate that they settle into an "approximate" splay state, and their phases are no longer uniformly spread. For $k = 1$, we examine the system's final state as a map and show that phase-locking synchronization can never occur for nonidentical oscillators. We also derive a sufficient and necessary condition for the existence of cycles.

**The study of synchronization has been extremely prominent for over two decades. The phenomenon is present in many systems in physics, biology, and engineering. The Kuramoto model, one popular system that models synchronization, describes $n \gg 1$ phase oscillators on a line. In this paper, we investigate a variant which was first proposed by Wiley, Strogatz, and Girvan [1]. This variant differs from the Kuramoto model by describing $n$ oscillators on a ring rather than a line topology. Our system generalizes theirs in that the oscillators are not necessarily identical. Furthermore, to better understand the final states that arise, we include an examination of the final state as a map between oscillators rather than a set of coupled differential equations.**

## Introduction

Over the past two decades, myriad researchers have spent a great deal of time studying synchronization, which combines ideas from nonlinear dynamics and network theory. *Synchronization* is the process by which interacting, oscillating objects affect each other's phases such that they spontaneously lock to a certain frequency or phase [2]. It arises in numerous areas of physics, biology, and engineering and can have either a positive or a negative impact; synchronization is required for the successful operation of many real-world systems, such as large populations of fireflies [3], the natural pacemakers of the heart [4], superconductors [5], laser light [6], and the natural circadian rhythms of the human brain [7]. However, it has also led to the downfall of some systems, such as the Millennium Bridge [8].

One of the canonical models that exhibits synchronization is the Kuramoto Model [9], which describes a population of $n \gg 1$ phase oscillators with internal frequencies drawn from some distribution:

$$\dot{\phi} = \omega_i + \sum_{j=1}^{n} \frac{K}{n} \sin(\phi_j - \phi_i). \tag{1}$$

Here $\phi_i$ represents the phase of oscillator $i$, $\omega_i$ the internal frequency, $K \geq 0$ the coupling strength, and the $\frac{1}{n}$ scaling ensures that the system is bounded as $n \to \infty$. This equation describes oscillators arranged on a line, but it can be modified for other topologies. In our work, for example, we consider a ring topology.

## The Model and Preliminaries

We investigate a generalization of the model studied by Wiley et al. [1]. Consider a system of $n \gg 1$ phase oscillators equally spaced on a ring, with initial conditions for phase $\phi_i$ $(i = 1, ..., n)$ randomly drawn from $[0, 2\pi]$ (see Fig. 1). In isolation, each oscillator would oscillate at its natural frequency $\omega_i$ (a feature not present in the Wiley paper [1]), but when in the proximity of other oscillators the rate of change in phase is affected by the phases of its $k$ nearest neighbors on either side of it.

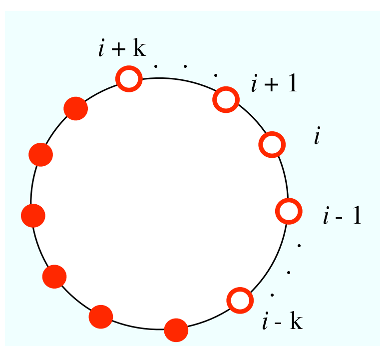



Figure 1: (Color online) A ring of $n$ equally spaced oscillators. The $i$th (for $i = 1, ..., n$) oscillator is coupled to its $k$ nearest neighbors on either side, shown here as hollow.

This system is described by the equation:

$$\dot{\phi}_i = \omega_i + \sum_{j=i-k}^{i+k} K \sin(\phi_j - \phi_i), \qquad i = 1, ..., n. \tag{2}$$

Note that the factor of $\frac{1}{n}$ is unnecessary because the sum is no longer infinite as $n \to \infty$.

Wiley et al. determined that when all the oscillators are identical (i.e., $\omega_i = \omega$ for all $i$) there are two types of attracting final states for the evolution of the oscillators' phases. These equilibria occur after some transient dynamics, that is, after the oscillators' phases converge to a final value (see Fig. 2a in which equilibrium is reached at approximately time = 800). One equilibrium represents the complete phase-locking of all phase oscillators. In the other, the system settles into uniformly-twisted traveling waves called "$q$-twisted" or "splay" states (note that the synchronized state is actually a $q$-twisted state with $q = 0$). In these splay states, oscillators are staggered equally in the phase space and move at the same final frequency. However, it is also possible in other applications for a splay state to occur with oscillators staggered equally in time [5]. The number $q$ is a winding number given by traveling around the circle of oscillators and counting the number of phase twists. Furthermore, Wiley et al. showed using heuristic arguments that the probability that random initial conditions will lead to a state which is $q$-twisted is a Gaussian distribution over $q$ with mean zero and standard deviation proportional to $\sqrt{n/k}$. We have reproduced these results (see Fig. 3).

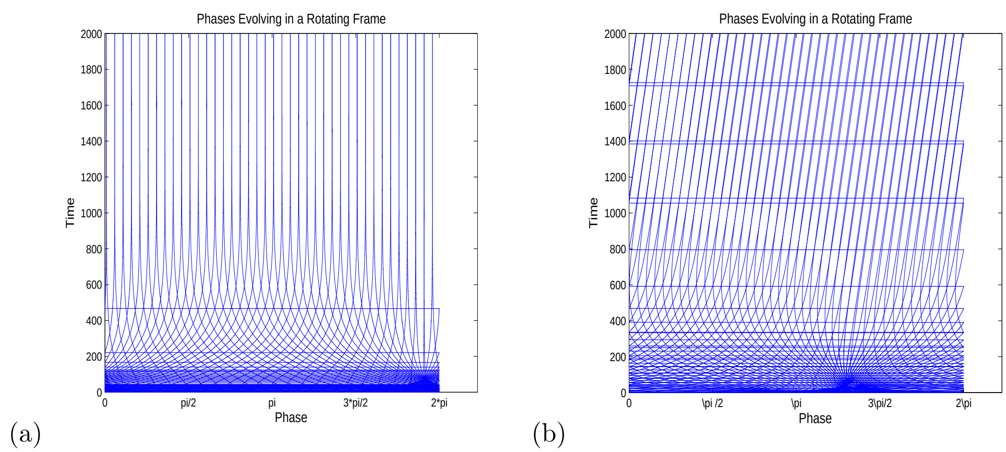
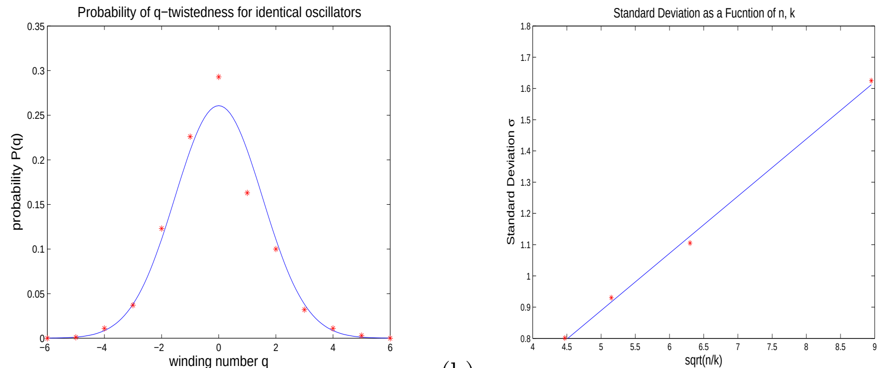



Figure 2: Plots of the phases of the oscillators evolving over time in a rotating frame. (a) A typical plot of time against phase for identical oscillators with $k = 1$, $n = 80$. In this case the system settles into $q$-twisted state with $q = 2$. (b) The system for oscillators with frequencies taken randomly and uniformly from the interval $[-0.1, 0.1]$ settles into $q = 2$ as well, but the phases are no longer uniformly spaced and the average frequency is nonzero.



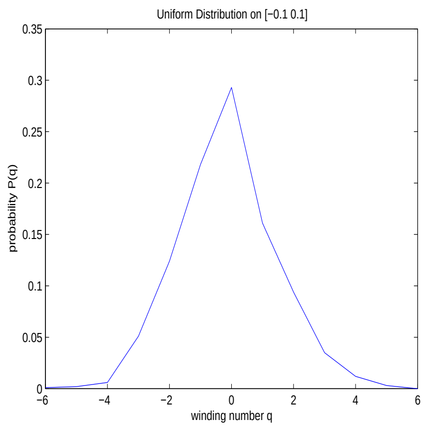

(a) (b)

Figure 3: (Color online) Results reproducing Wiley et al.'s research for identical phase oscillators. (a) The probability that the system reaches a $q$-twisted state is distributed approximately normally with respect to $q$ for identical oscillators. The results from the numerical simulation (for 1000 random initial conditions, with $k = 1$, $G = 1$, and $n = 80$) are given by the plotted stars, and the Gaussian with the standard deviation of the data is shown by the curve. (b) Numerics suggest that standard deviation $\sigma$ is proportional to $\sqrt{n/k}$ for $G = 1$. Data is given by the stars, and linear regression gives $\sigma \approx 0.19\sqrt{n/k}$. We also show the line that fits the data.

# Numerics

In their paper, Wiley et al. assumed a delta-function distribution–that is, all of the oscillators have the same internal frequency. We first reproduced their results, confirming that the probability that the final state is $q$-twisted is approximately Gaussian (see Fig. 3a). Wiley et al. then used the least squares method to fit the data to a one-parameter (the standard deviation $\sigma$) discretized Gaussian distribution using the least squares method, and found $\sigma$ to be linearly proportional to $\sqrt{n/k}$. We approximated the standard deviation directly from the data, recreating the linearity of $\sigma$ to $\sqrt{n/k}$ (see Fig. 3b).

We took the natural frequencies of the oscillators from a uniform distribution over a small interval centered at zero with the intent to later generalize to various other distributions. There is no loss of generality from the center being at zero because one can always put the system into a rotating frame. As the size of the distribution interval increases, the oscillators develop a nonzero average frequency, and they are no longer uniformly spread apart in phase. Instead, they miss the $q$-twisted states with which they are associated by a small amount which increases with interval width (see Fig. 2). Furthermore, numerical experiments show that as interval width increases, the standard deviation decreases and the absolute value of the mean increases (see Fig. 4).

(a) 

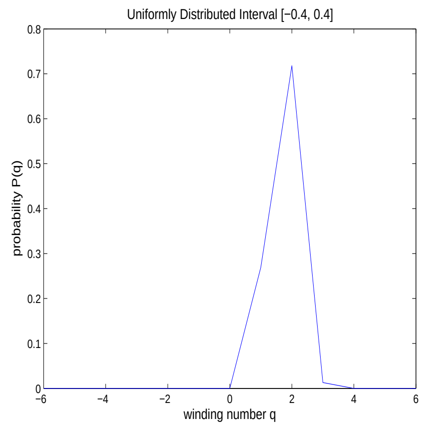

(b) 


Figure 4: Probability distributions for nonidentical oscillators. (a) Distribution for oscillators with internal frequencies taken randomly from a uniform distribution on $[-0.1, 0.1]$ with $k = 1$, $n = 80$, and 1000 random initial conditions. (b) The distribution for oscillators with internal frequencies taken randomly from a uniform distribution on $[-0.4, 0.4]$ with $k = 1$, $n = 80$, and 1000 random initial conditions.

# Discrete Formulation

To better understand the approximate $q$-twisted states, we study the analytics from a different point of view. Thus far, we have been looking at the system in terms of how each oscillator changes phase over time, but now we will switch to studying how each oscillator acts in response to its neighboring oscillators. As an analogy, if one were to study the dynamics of a crowd of people entering an auditorium, instead of looking at each person's evolution in time, we would look at how each person moves around their neighbors–if everyone around him is moving towards the entrance, he will move in the same direction. That is, we use our equations to arrive at a discrete formulation of the system as follows.

We let $k = 1$ (when $k > 1$ the map becomes much more complicated and has higher dimensionality). Assume that the oscillators have reached their equilibria and are at their final state with common frequency $\Omega$. Substituting into (1) yields

$$\Omega = \omega_i + \sin(\phi_{i+1} - \phi_i) + \sin(\phi_{i-1} - \phi_i). \tag{3}$$

Let $v_i = \phi_{i+1} - \phi_i$ and $P_i = (\Omega - \omega_i)/G$. Rewriting the equation, we thereby obtain the following map:

$$\begin{aligned} \phi_{i+1} &= v_i + \phi_i \,, \\ \sin v_{i+1} &= P_{i+1} + \sin v_i \,. \end{aligned} \tag{4}$$

It should be noted that when taking the arcsine there are two possible values for $v_i$. However, this map sends $v_i$ to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore this does not pose a problem if $v_i$ is sufficiently small, or if adjacent oscillators are sufficiently close in phase. In the original dynamical system (1), once the oscillators settle into a final state with phases uniformly distributed from 0 to $2\pi$. Because we assumed that $n$ is large, the difference between adjacent oscillators must be small (in fact, smaller than $\frac{\pi}{2}$). For adjacent oscillators to be $\pi/2$ apart in phase, the winding number must be $q = n/4$. For a system with $n = 80$, this corresponds to a winding number of $q = 20$: however, the probability of this occurring is very small.

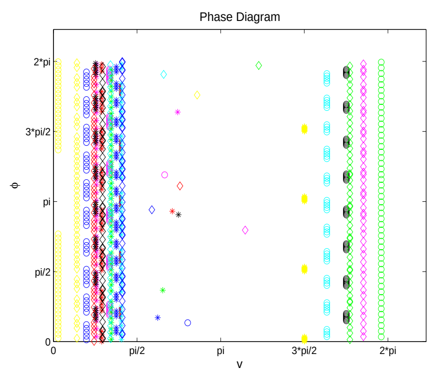



Figure 5: (Color online) Phase diagram for $P_i = 0$. Different symbols represent the iterations of (4) for different initial conditions. The phases of adjacent oscillators jump around the phase space by a fixed amount while $v$ stays constant (i.e., $v_i = v$ for all $i \geq 2$). Here, we plot 28 different random initial conditions.

When the oscillators are identical, $P_i$ is identically zero. In this case, $v_i$ is constant for all $i$, so subsequent $\phi_i$ values simply jump around the phase space by a fixed amount, corresponding to the phases being uniformly spread apart (see Fig. 5). Because $\phi_i$ is a periodic variable, it is possible that some iteration may exactly fall upon a previous iteration, leading to a cycle in the map. This only occurs for values of $v_i$ for which there exists a solution for the following equation: $2\pi m_1 = m_2 v_i$, where $m_1$, $m_2$ are integers and $v$ is the constant phase shift ($v = v_i$ for all $i \geq 2$). Hence, the existence of such cycles depends solely on the value of $v_1$.

When $P_i$ is constant and positive, solutions follow a family of curves until $v_i = \frac{\pi}{2}$ ($v_{i+1}$ is then complex). There are no cycles in this case (see Fig. 6ab). When $P_i$ is constant and negative, the real portion of the phase plane rotates by 180 degrees around the origin (see Fig. 6cd).

We have also determined a necessary and sufficient condition for the existence of cycles for general $P_i$: a $K$-cycle exists if and only if

$$\sum_{j=i+1}^{i+K} P_j = 0 \bmod(2\pi) \quad \text{and} \quad \sum_{j=i}^{i+K-1} v_j = 0 \bmod(2\pi) \tag{5}$$

for all $i$. We confirmed these results using numerical simulations. In particular, this condition shows that full synchronization (corresponding to $K = 1$) occurs if and only if $P_i = 0$ (mod($2\pi$) and $v_i = 0$ (mod($2\pi$), or, when $P$ and $v$ are identically zero. Therefore, exact phase-locking only occurs when oscillators are identical. Hopefully (5) will help to shed more insight on the required distribution for $\omega_i$ to have precise $K$-cycles.

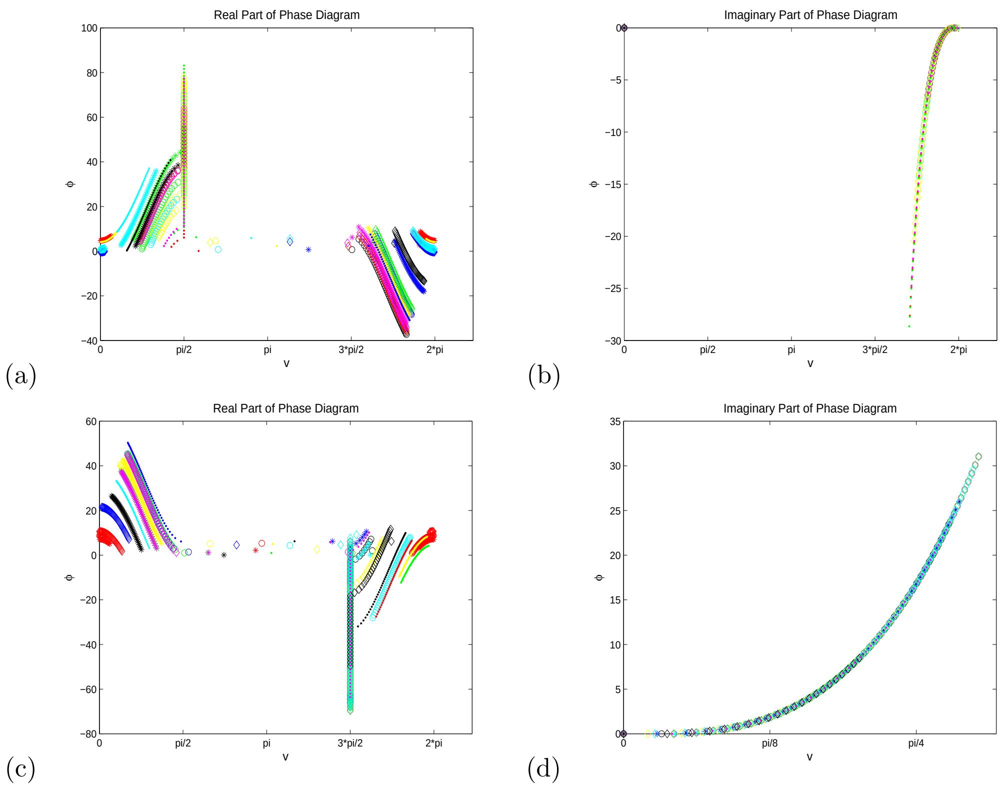



Figure 6: (Color online) Phase diagrams for constant but nonzero $P_i$. Note that in these diagrams $\phi$ is not for all of $\mathbb{R}$ for clarity. (a) The real part of the phase diagram for $P_i = 0.1$. (b) The imaginary part of the phase diagram for $P_i = 0.1$. (c) The real part of the phase diagram for $P_i = -0.1$. (d) The imaginary part of the phase diagram for $P_i = -0.1$.

# Conclusions and Further Work

We studied the Kuramoto model on a ring topology (2) and found that when oscillators are no longer identical, they no longer fall into the conventional notion of a splay state. We rewrote (2) as a map (4) (with $k = 1$) to further study the system. For $k = 1$, complete phase-locked synchronization only occurs when the oscillators are identical. In addition, we have found a necessary and sufficient condition (5) for the existence of cycles in the map (4).

The next step in this project is to connect the results obtained by examining the map (4) to the dynamical system (2), especially the condition for cycles. Possible future work includes using different distributions to obtain the natural frequencies. For example, one might examine the case where the frequencies are distributed in a near-delta distribution (that is, one where most oscillators are at the same frequency but a few are not). One could also study the synchronization phenomenon in different networks or topologies. Instead of a simple ring, one might consider having "shortcuts" such that diametrically opposed oscillators are coupled. Alternatively, instead coupling at the same strength, one might consider a gradual decay in coupling as distance from the oscillator increases rather than only being coupled to the $k$ nearest neighbors.

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