The Dynamics of Pendula: An Introduction to Hamiltonian Systems and Chaos

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Abstract

The Laser Interferometer Gravitational Wave Observatory (LIGO) was built to detect and observe gravity waves, whose existence was predicted by Einstein's theory of general relativity. A critical part of the interferometer are the mirrors which reflect the laser beams, and whose motion determines if a gravity wave is present. As of 2001, LIGO was slated to update their facility by housing each mirror on the lowest bob of a quadruple pendulum to reduce thermal noise.

With this system as motivation, I examine simple and multiple-component pendula and use them to convey basic techniques to study Hamiltonian systems. New terminologyand concepts will be introduced when relevant to a perticular pendulum system. By increasing the complexity of the system, the material covered will increase in difficulty. Some of the concepts covered will include: stability, Runge-Kutta integration, nonautonomous systems, Euler-Lagrange Equations, and Poincaré sections.

1 Introduction

Nonlinear dynamics is the study of time-evolving systems governed by equations where superposition fails [13]. Numerous analytical and numerical techniques have been developed to analyze such dynamical systems.

Even elementary systems introduced in one's first physics class are nonlinear in nature. For example, consider the unforced simple pendulum described by the following differential equation:

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0, \tag{1}$$

where g is the constant gravitational acceleration, L is the length of the pendulum, and θ is the displaced angle. One uses a small angle approximation ($\sin x \sim x + O(x^3)$ for small x) to approximate the motion of the simple pendulum as linear. This simplification eliminates several solutions to a pendulum's motion such as when a pendulum whirls completely around its pivot [13]. The exact analytical solution of (1) can be expressed in terms of Jacobian elliptic functions [3], but one can gain a considerable amount of insight by instead using qualitative methods.

1.1 Hamiltonian Systems

Hamiltonian systems comprise a class of dynamical systems in which some quantity (typically energy) is constant along the system's trajectories [3]. Such systems take the form $H(x, p) =$ constant for some set of displacements and momenta. The associated dynamical system is given by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial x}, \end{aligned} \tag{2}$$

which are termed *Hamiltons's Equations*. The quantity H is called the *Hamiltonian*. If the conserved quantity is energy, it is given by the sum of the potential and kinetic terms. Hamiltonian systems have an equal number of positions and

momenta that describe the motion of the system. Each position-momentum pair comprises a *degree of freedom* (dof). The Hamiltonian structure of a system can be exploited when studying dynamics [16].

Undamped pendula are canonical examples of Hamiltonian systems. The dynamics of multiple-component pendula are of great practical interest, as evinced by GEO 600, a gravity wave detection facility in Germany that uses triple pendula to reduce thermal noise in their interferometers [12]. Additionally the Laser Interferometer Gravitational Wave Observatory (LIGO) was built to detect and observe gravity waves, whose existence was predicted by Einstein's theory of general relativity. A critical part of the interferometer are the mirrors which reflect the laser beams, and whose motion determines if a gravity wave is present. As of 2001, LIGO was slated to update their facility by housing each mirror on the lowest bob of a quadruple pendulum to reduce thermal noise [11]. These developments have produced scientific interest in the dynamics of triple pendula [15, 4, 14].

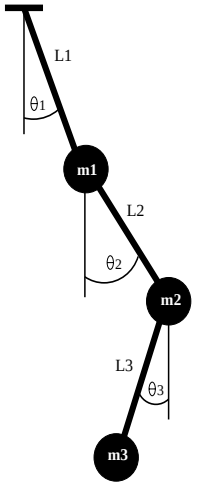

Figure 1: Triple pendulum with mass (m_1, m_2, m_3), length (L_1, L_2, L_3), and angle $(\theta_1, \theta_2, \theta_3)$ labeled.

To be described as chaotic, a system must satisfy three general criteria [13]: the motion of the system must be extremely (i.e., exponentially) sensitive to the

initial conditions, as small input errors can produce drastically different outputs. It is impossible to predict the motion of a chaotic system after a short period of time because of such sensitivity. Second, chaotic systems are recurrent, in that if one defines a neighborhood (no matter how small) around an initial point in phase space, the trajectory starting at that point will intersect that neighborhood infinitely many times during its time-evolution. Third, trajectories do not settle down. As t goes to ∞ the trajectory does not become a fixed point, periodic orbit, or a quasi-periodic orbit [13].

2 Simple Pendulum

We begin our discussion with some terminology from nonlinear dynamics. In the autonomous system $\dot{x} = f(x)$, the *phase space* is the space with coordinates $(x^1, ..., x^n)$, which is is n-dimensional [13]. (Autonomous means the system is not dependent on time). The *trajectories* are solutions of the system plotted in the phase space starting at an initial condition $x(0)$. Along them, the Hamiltonian is constant. *Equilibria* occur at points in phase space that satisfy $f = 0$, they are called equilibria because if a trajectory begins at an equilibrium it will remain there for all time. A plot of all the qualitatively different trajectories is called a *phase portrait* [13]. Equilibria determine the appearance of the phase portrait.

The simple pendulum has one dof. With Newtons laws, one can derive the equations of motion for the simple pendulum (1).

2.1 Linearization about the Equilibria

Linearization determines the local stability of the equilibrium by computing if a small disturbance from the equilibrium grows or decays [13]. This is done by computing the Jacobian matrix for the system at the specified equilibrium. For the simple pendulum, we begin by writing equation (1) as a pair of first order differential equations

$$\dot{\theta} = v,$$
$$\dot{v} = -\sin\theta. \tag{3}$$

The Jacobian for the simple pendulum is

$$\begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}. \tag{4}$$

With the Jacobian, a new matrix $\mathbf{A}$ can be computed for each of the equilibria. The equilibria for the simple pendulum are $(0,0)$ and $(\pi,0)$. The matrix $\mathbf{A}$ is the Jacobian matrix solved at one of the equilibria. So at $(0,0)$, $\mathbf{A}$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

From this matrix we can compute the eigenvalues using the *characteristic equation* $\det(\mathbf{A} - \gamma I) = 0$, where I is the identity matrix. (Solving for γ will yeild the eigenvalues). At $(0,0)$, $\gamma = \sqrt{-1}$ so the eigenvalues are complex. While at $(\pi,0)$, $\gamma = 1, -1$. (Computation of $\mathbf{A}$ at $(\pi,0)$ is left for the readers). Then, after solving for the eigenvalues of this matrix the local stability of the equillibria can be determined depending on the values of the eigenvalues.

2.2 Stability of Equilibria

The two equilibria of the simple pendulum occur at the apex and the lowest point of the pendulum's motion. In the previous section, we found the eigenvalues at the origin (the lowest point of the pendulum's swing) to be complex. Complex eigenvalues occur when the equilibrium is a center or a spiral. The distinction is made when the eigenvalues are pure imaginary, when the *trace* is equal to zero. (The trace is the addition of the diagonal entries of the matrix). Otherwise the equilibria is a spiral. The bottom position of a simple pendulum is a *center*, for which trajectories nearby are neither attracted nor repelled and remain in periodic orbits nearby. An equilibrium that is a center is stable. To

find the direction of the flow compute a few vectors in the vector field [13]. The rotation about the equilibria located at the origin (a center) is clockwise.

The pendulum has another equilibrium at $\theta = \pi$ (the top of its motion). The eigenvalues at this equilibrium are positive and negative. This equilibrium is a *saddle*, so trajectories are both attracted and repelled. Saddles are unstable because of this. With this information and the energy equation (6) of this system,

$$\frac{1}{2}\dot{\theta}^2 - \cos\theta = E, \tag{6}$$

where $\dot{\theta}$ is the velocity, and E = constant, one obtains a complete qualitative description of the system's dynamics (see Figure 2).

2.3 Numerical Analysis of the Simple Pendulum

Numerical integration must be used to complement analytical work because differential equations can almost never be exactly solved analytically. Using the Runge-Kutta method [7], we can simulate the single pendulum using Matlab (Figure 2).

The phase space for the simple pendulum is cylindrical because θ is a periodic variable. The different saddle points in the phase plane represent the same state, as indicated by the cylindrical phase space. They are all unstable equilibrium at the apex of the pendulum's motion. (When the pendulum is positioned straight up, any disturbance will cause the pendulum to be repelled from the equilibrium and swing from the apex).

The trajectory which intersects the saddle forms a *separatrix*; on either side of this trajectory the motions are qualitatively different. Trajectories outside the separatrix correspond to periodic whirling motion, while trajectories inside of the separatrix are librations. *Librations* are small oscillations about an equilibrium. This is the motion commonly identified with a pendulum: the oscillations about the origin.

In a cylindrical phase space, the separatrix becomes a *homoclinic orbit* where

the trajectory begins and ends at the same point in phase space. Homoclinic orbits are common in Hamiltonian systems, but rare in non-conservative systems. (The separatrix would be a pair of *heteroclinic orbits* if the phase space was planar, two trajectories beginning and ending at equilibria).

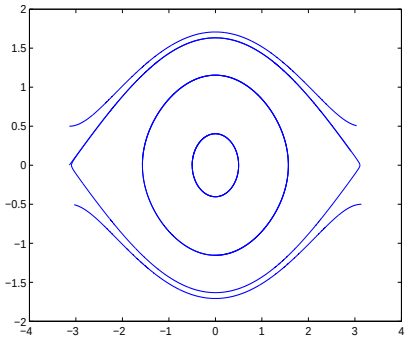

Figure 2: Phase portrait of the simple pendulum: a plot in the plane $(\theta, \dot{\theta})$ where θ is from $(-\pi, \pi]$. Theta is a periodic variable. The phase space is divided into two distinct types of motion by the separatrix: at high energies the pendulum whirls over the top (outside the separatrix), while at low energies the pendulum oscillates back and forth.

3 Forced Simple Pendula

3.1 Nonautonomous Systems

The forced simple pendulum is an example of a nonautonomous system, or a time-dependent system. This is because the forcing function is dependent on time. It consists of a simple pendulum which is perturbed in some direction, in this case horizontal. The new Hamiltonian H will simply be the addition of the old Hamiltonian and a perturbation in the horizontal direction.

$$H = H_0 + \epsilon \cos(\omega t), \tag{7}$$

where ϵ is the amplitude of forcing, ω is the freqency of forcing, and t is time. There is also parametric forcing which corresponds to forcing in the vertical direction. The equation for a pendulum being shaken up and down is

$$H = H_0 + \epsilon \sin(\omega t), \tag{8}$$

but we will focus on horizontal forcing.

The forced simple pendulum has 2 dof, time adds another dof. Actually there is still much mathematical debate about the number of degrees of freedom. Many people say that the system has $1\frac{1}{2}$ degrees of freedom because time is not an entire position and momentum pair and therefore cannot constitute an entire degree of freedom [5]. With time dependence, the forced simple pendulum is represented as a system of three first order differential equations (9).

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= \frac{-g}{L} \sin\theta + \epsilon \cos(\omega t) \\ \dot{t} &= 1, \end{aligned} \tag{9}$$

where θ is the displaced angle, v is the velocity, g is the constant gravitational acceleration, L is the length of the pendulum, ϵ is the amplitude of forcing, and ω is the freqency of forcing. The third equation could also be defined as $z = \omega t$.

3.2 Numerical Analysis

To understand the forced simple pendulum system (9), it is essential to simulate it numerically. This can be done by using the Runge-Kutta method[7]. The Runge-Kutta method is a fourth-order integration method. Basically, the Runge Kutta method extrapolates each successive point over a discrete time interval. This method is prefered by many scientists because of its simplicity and accuracy [13].

No phase portrait can be derived from the trajectories calculated because no general qualitative behavior applies to any one area in the phase plane. For

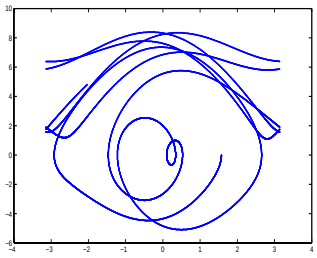

Figure 3: This is a sample trajectory from a Runge-Kutta simulation where $x(0) = \frac{\pi}{2}$, $y(0) = 0$, $\omega = 3$, $\epsilon = 5$, and run from $t = [0, 15]$ with a time step of $\delta t = .001$.

example, with the simple pendulum, the separatrix separated rotations and librations. Another method must be used to numerically analyze the motion of the forced simple pendulum.

3.3 Poincaré Sections

The numerical analysis of the forced simple and double pendulum will involve some new concepts, including Poincaré sections, which are created as follows. Whenever a trajectory meets some "stopping condition," all variables (aside from that used to determine the condition) are plotted. This is repeated for qualitatively different trajectories. The resulting Poincaré section is a map rather than a vector field; it has one dimension less than the phase space. A useful analogy is to imagine that a piece of paper is inserted into the phase space so that the trajectory intersects with it [6]. At such intersections, one marks the paper; repeating this procedure for qualitatively different trajectories yields a Poincaré section. Poincaré sections are useful when analyzing chaotic systems [13], as they make it easier to understand their dynamics.

It is very important to construct Poincaré sections intelligently to be able to visualize the dynamics of chaotic systems. For example, with periodically forced single pendula, one defines a Poincaré section by plotting the location of trajectories for $t = nT$, where T is the forcing period and $n > 1$ is an integer.

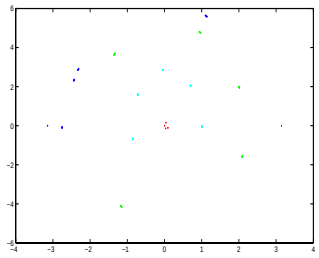

Figure 4: This plot is shows many places where the forced simple pendulum exhibits periodic behavior. The axises are (θ, v).

In comparison a Poincaré section of a simple pendulum looks exactly like the phase portrait, this is why Poincaré sections are not necessary for non-chaotic systems.

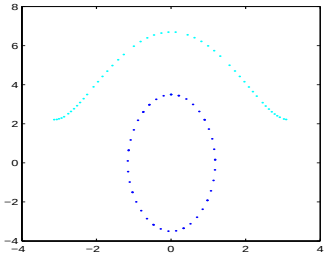

Figure 5: In comparison to the other figure, this plot is very similar to the phase plane of the simple pendulum no extra information can be garnered by using Poincaré sections on non-chaotic systems.

4 Double Pendula

For the simple pendulum, the equations of motion were derived using force balance. For the forced simple pendulum, the equations of motion were derived by adding a perterbation to the Hamiltonian of the simple pendulum. The double pendulum is more complex and force balace would quickly become rigorous.

Instead we can derive the equations of motion for the double pendulum using energy balance producing the equations of motion using the Euler-Lagrange equations.

4.1 Euler-Lagrange Equations

The *Euler-Lagrange* equations [3] can be used instead of force balance to derive equations of motion. *Lagrangian* and *Hamiltonian* descriptions are complementary ways to describe physical systems, as one can convert from one to the other with a Legendre transform [8]. Using the Euler-Lagrange equations we can derive the equations of motion for the double pendulum by initially defining the positions of the first and second pendulum bobs:

$$\begin{aligned} x_1 &= L_1 \sin(\theta_1), \\ y_1 &= L_1 \cos(\theta_1), \\ x_2 &= L_1 \sin(\theta_1) + L_2 \sin(\theta_2), \\ y_2 &= L_1 \cos(\theta_1) + L_2 \cos(\theta_2), \end{aligned} \tag{10}$$

where (x_1, y_1) and (x_2, y_2) represent the cordinates of the pendulum bob in a plane, L_1 and L_2 are the lengths of the arms, and θ_1 and θ_2 are the displaced angles. The velocities are the derivatives of the position equations:

$$\begin{aligned} \dot{x}_1 &= \dot{\theta}_1 L_1 \cos(\theta_1), \\ \dot{y}_1 &= -\dot{\theta}_1 L_1 \sin(\theta_1), \\ \dot{x}_2 &= \dot{\theta}_1 L_1 \cos(\theta_1) + \theta_2' L_2 \cos(\theta_2), \\ \dot{y}_2 &= -\dot{\theta}_1 L_1 \sin(\theta_1) - \theta_2' L_2 \sin(\theta_2). \end{aligned} \tag{11}$$

To understand this, the triple pendulum in Fig. 1 provides a good guide. These values are then used to compute the kinetic (T) and potential (V) ener-

gies.

$$T = \sum_{n=1}^{2} \frac{1}{2} m_n v_n^2$$
$$V = \sum_{n=1}^{2} g m_n h_n \tag{12}$$

where m_i is the mass of each pendulum bob, v_i is the velocity of each bob, g is the constant gravitational acceleration, and h_i is the distance from the local zero potential of each bob [10]. The kinetic and potential energy are then used to compute the *Lagrangian*,

$$L = T - V, \tag{13}$$

which gives the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \theta_n'}\right) - \frac{\partial L}{\partial \theta_n} = 0, \quad n \in \{1, 2\}. \tag{14}$$

The resulting equations of motion are

$$(m_1+m_2)L_1^2\theta_1'' + m_2 L_1 L_2 \theta_2'' \cos(\theta_1-\theta_2) + m_2 L_1 L_2 \theta_1' \theta_2' \sin(\theta_1-\theta_2) + g L_1 \sin\theta_1 (m_1+m_2) = 0$$

$$m_2 L_2^2 \theta_2'' + m_2 L_1 L_2 \theta_1'' \cos(\theta_1-\theta_2) - m_2 L_1 L_2 \theta_1'^2 \sin(\theta_1-\theta_2) + g m_2 L_2 \sin\theta_2 = 0. \tag{16}$$

4.2 Nondimensionalization

An important method of simplifying systems of equations is *nondimensionalization*, which reduces the number of parameters by creating dimensionless ratios [2]. Nondimentionalization makes it easier to determine which combinations of parameters are important to understand the qualitative behavior of dynamical systems. For the double pendulum, he dimensionless terms M and l are defined as follows:

$$M = \frac{m_2}{m_1 + m_2}$$

$$l = \frac{L_2}{L_1} \tag{18}$$

Methodically replacing terms with dimensionless ratios, and substituting $\omega^2 = g/L_1$ results in simplified versions of the original system of differential equations,

$$\theta_1'' + Ml\theta_2'' \cos(\theta_1 - \theta_2) + Ml\theta_1'\theta_2' \sin(\theta_1 - \theta_2) + \omega^2 \sin\theta_1 = 0$$

$$l\theta_2'' + \theta_1'' \cos(\theta_1 - \theta_2) - \theta_1'^2 \sin(\theta_1 - \theta_2) + \omega^2 \sin\theta_2 = 0, \tag{20}$$

and also indicates which parameter combinations are important. Next, substitute $\phi = (\theta_1 - \theta_2)$ to write (20) as a dynamical system,

$$\begin{aligned}
\theta_1' &= v_1 \\
\theta_2' &= v_2 \\
v_1' &= \frac{M\omega^2 \sin\theta_2 \cos\phi - Mv_1^2 \sin\phi\cos\phi - Mlv_1v_2 \sin\phi - \omega^2 \sin\theta_1}{1 - M\cos^2\phi} \\
v_2' &= \frac{Mlv_1V_2 \sin\phi\cos\phi + \omega^2 \sin\theta_1 \cos\phi + v_1^2 \sin\phi - \omega^2 \sin\theta_2}{l - Ml\cos^2\phi}.
\end{aligned} \tag{21}$$

4.3 Analytics for the Double Pendulum

The same analytical techniqes used for the unforced simple pendulum can also be applied to study the double pendulum. However, the process quickly becomes complex and can require the use of computer algebra software. However the equilibria can be determined algebraically.

The equilibria occur when $v_1, v_2 = 0$, $\theta_1, \theta_2 = \frac{k+\pi}{2}$ where k is an integer from $[0, 2]$, and when $\theta_1 = \pi + \theta_2$. So the four equilibria are:

1. $(0, 0, 0, 0)$
2. $(\frac{\pi}{2}, \frac{\pi}{2}, 0, 0)$
3. $(\pi, \pi, 0, 0)$
4. $(\pi, 0, 0, 0)$

Next, use linearization to calculate the stability of the equilibria. The linearization for the double pendulum is more complex due to the increased number

and complexity of the equations of motion. For example, the Jacobian matrix for the double pendulum is 4×4, while for the unforced simple pendulum it was 2×2. To compute the eigenvalues one must compute the determinant of a 4×4 matrix, on how to do this see [1].
At this point, it becomes necessary to use computer algebra software. However, this article is not expositing computer algebra software, so we will pursue an alternate method of determining the stability of the equilibria.

One can also find stabilities numerically. For example, by looking at the phase portrait of the simple pendulum once can easily see that the origin is a stable equilibrium.

4.4 Numerics for the Double Pendulum

The double pendulum is a system of four differential equations and has 2 dof. The phase space for the double pendulum is four-dimensional. Despite this, the Runge-Kutta method can be used. However, it is difficult to visualize what is happening because no graph produced by Matlab can capture all the information. One way to visualize the data is by producing four different graphs: (a) θ_1 v. θ_2, (b) v_1 v. v_2, (c) θ_1 v. v_1, and (d) θ_2 v. v_2. Figure (6) below is an example.

Since looking at four graphs at once and deriving useful information from them is difficult and tedius, we will employ another method to study double pendula.

4.5 Poincaré Sections

For the unforced double pendulum, one can make a Poincaré section by plotting the location of the trajectories when $\theta_1 = 0$. This results in a three dimentional Poincaré section. For simplicity's sake I eliminated one variable from the maps I produced.

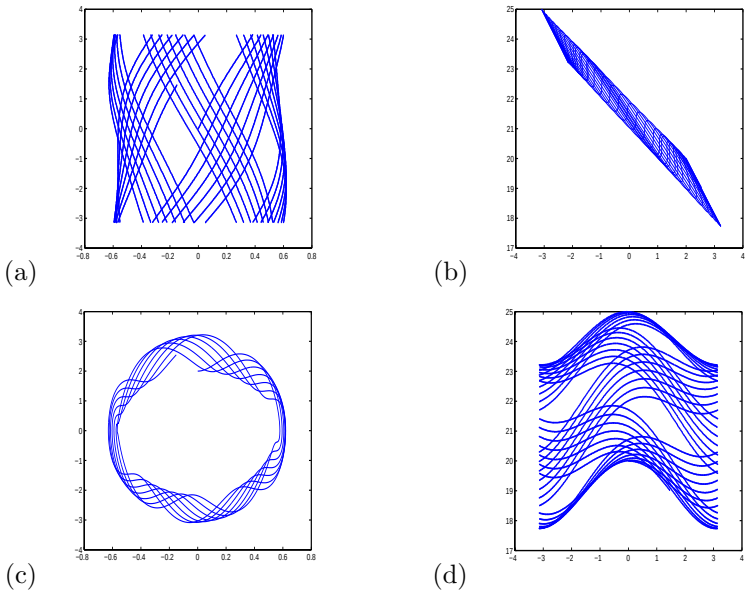

Figure 6: This is a compellation of trajectories from a simulation from the double pendulum. It is difficult to see what the system is doing as a whole, but graph (c) representing the top bob has a quasi-periodic orbit about its origin, while graph (d) representing the second bob shows that it is whirling. This graph had the initial conditions: $\theta_1 = 0$, $\theta_2 = 0$, $v_1 = 2$, $v_2 = 20$, and run for $10,000$ iterations with a step-size of .001.

Future Considerations

4.6 Triple Pendulum

The equations of motion for the triple pendulum can be derived using the Euler-Lagrange equations. It is actually the most convenient method especially since the equations become large.

The triple pendulum has three degrees of freedom. Poincaré sections can be used to analyze the motion of the triple pendulum, however phase space analysis and perturbation theory are more common techniques when studying chaotic systems [9]. Also common are perturbative techniques that include averaging

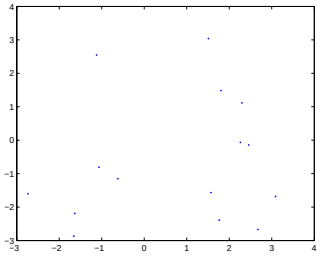

Figure 7: When representing chaotic behavior Poincaré sections resemble scatter plots, with enough iterations, they begin to define and encircle areas that are non-chaotic and periodic areas.

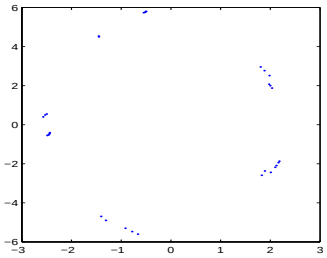

Figure 8: This is a plot of quasi-periodic areas on the double pendulum's phase plane.

and the method of multiple scales. To continue studying chaos and Hamiltonian systems the autor reccomends researching these methods.

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