The Two-Colour Theorem

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Easy Questions, Difficult Answers?

Many problems in graph theory can be explained in a few minutes, and in a way that those with no mathematical background, including young children, can understand. Examples include the bridges of Koenigsberg (Fig. 1) and the Water, Gas and Electricity Puzzle (Fig. 2). Both of these puzzles are impossible, and this can be shown by simply systematically exhausting all of the possible solutions. They both hint at deeper, more general theorems, though. The bridges of Koenigsberg is an impossible puzzle, because for a graph to have an Euler cycle (a circuit that runs along every edges exactly once and returns to its original starting point) it is a necessary and sufficient condition that every node has even degree. In this case, none of the nodes have even degree. Proving the *necessary* part of this theorem is easy: for every time that an island/node is “entered” it needs to also be “exited”. The *sufficient* part of the theorem is harder. The Water, Gas, Electricity puzzle is impossible, because $K_{3,3}$ (the complete bipartite graph on two sets of three vertices) is not planar. A planar graph is a graph that can be drawn on a plane in a way such that no edges cross each other. Testing whether or not graphs are planar is a well-studied, complicated problem in computer science.

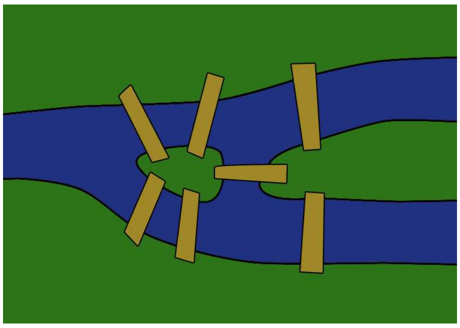

Figure 1: The bridges of Koenigsberg. Is it possible to create a walk that crosses each bridge exactly once, and also returns to its starting point?

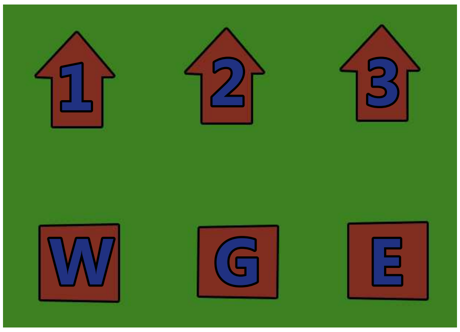

Figure 2: The Water, Gas and Electricity Puzzle. Is it possible to connect every house to all of water, gas and electricity, with no lines crossing?

The Four-Colour Theorem

When it comes to graph colouring problems, again, most problems are easy to explain, even though they they range between very easy and extremely hard to solve. Those who have heard of graph colouring problems will undoubtedly have heard of the four-colour theorem. The problem is easy to state in a way that even young children can understand. The question is the following: how many colours do we need to colour a map, such that two countries that share a border are not coloured with the same colour? It turns out that maps can be represented as planar graphs and vice versa, so the problem is the same on both objects. After trying to draw a few worst-case maps, most people will find maps that need 4 colours (Fig. 3), but no one will succeed at finding maps that need more than that. So, a natural conjecture is that there no such maps. Even though this seems like a very simple problem, it is very, very hard to prove. Although it has been around since the late 19th century, the conjecture was not proved until the 1970's, by Kenneth Appel and Wolfgang Haken, but it required exhaustive computer searches and hundreds of pages of analysis. There is no "pretty" proof for this problem (yet).

The Two-Colour Theorem

A more simple variation of the four-colour theorem is the two-colour theorem. This problem is a lot more than twice as easy, in fact! We look at maps that have a special rule: borders cannot end. This means that every border has to either go off the page on both ends, or it has to join up to itself in a cycle. To make things simpler (even though this is not necessary for the proof) we do not allow borders to cross themselves, and borders can cross, but they cannot lie "on top" of each other. Fig. 4 shows an example of this type of special map, which we will call an IB-map (infinite border map).

By trying to colour in such maps, most people will find that they need only

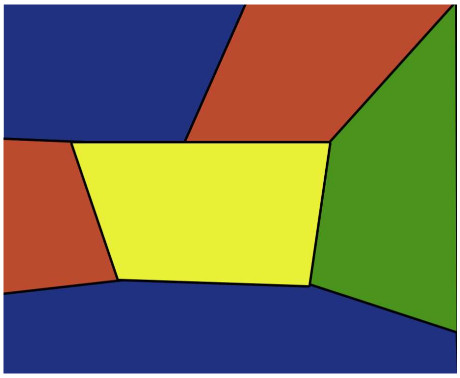

Figure 3: This map needs four colours to be coloured in properly, but are there maps that need more than that?

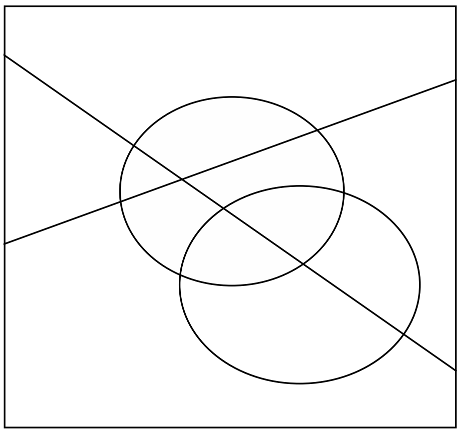

Figure 4: An infinite border map.

two colours. So, a natural conjecture is that all IB-maps can be coloured using only two colours. Is this as hard to prove as the four colour theorem? No! In fact, this is quite easy to prove, using a method called "proof by induction".

Proving the Two-Colour Theorem

Proof by induction, simply put, is a method of proving that statements are true for all natural numbers $(1, 2, 3, ...)$, by first proving that it is true for $n = 1$, and then proving that if it is true for some $n = k$, then is it true for $n = k + 1$. In other words, if we start at 1, and it is always okay to add 1, then we can reach all natural numbers $1, 2, 3, ...$ up to infinity. In this case, we will let n be the number of borders in out map. We will show that we can colour a map with one border, and then we will show that if we have a map with some number of borders k, and it has a proper two-colouring, then we can add a border and give it a new, proper two-colouring. The important thing to notice about IB-maps, is that every border splits the map into exactly two pieces. These pieces only meet each other on the border, and nowhere else. In other words, there is no way to walk around the border and end up on the other side without ever crossing it. This makes it easy to colour the map where $n = 1$. If we are colouring using only the colours black and white, then the one border splits our map into exactly two pieces, which we can colour black and white, respectively. Done! Now, we need to prove that if we have an IB-map that is coloured with the colours, we can add a border, and then "fix" the colouring. When we add a border to an already coloured map, then this border cannot lie on top of another border, so therefore at every point on the border, the two sides of it will have the same colour. This is because these two sides previously belonged to the same country. So, there is no point in this border where we are happy about the colouring, but everywhere else, we are. Here comes the clever step. We fix the colouring by inverting the colours on *one* side of the new border only. Inverting the two colours in an area does not make a good colouring bad anywhere in that area, so the colouring in the inverted area (one side of the new border) is still good. The colouring in the non-inverted area (the other side of the new border) is also still good, as we did not make any changes there. Everywhere on the new border, we started out with the two sides of the border having the same colour, but then we inverted all the colours on one side. So, now we have that everywhere on the new border the colouring is good. As an example, see Fig. 5, where we use this method to draw and colour the map from Fig. 4. We have now shown that any IB-map can be coloured using only two colours. This was not very difficult to prove, especially compared to the four colour theorem, which sounds so similar.

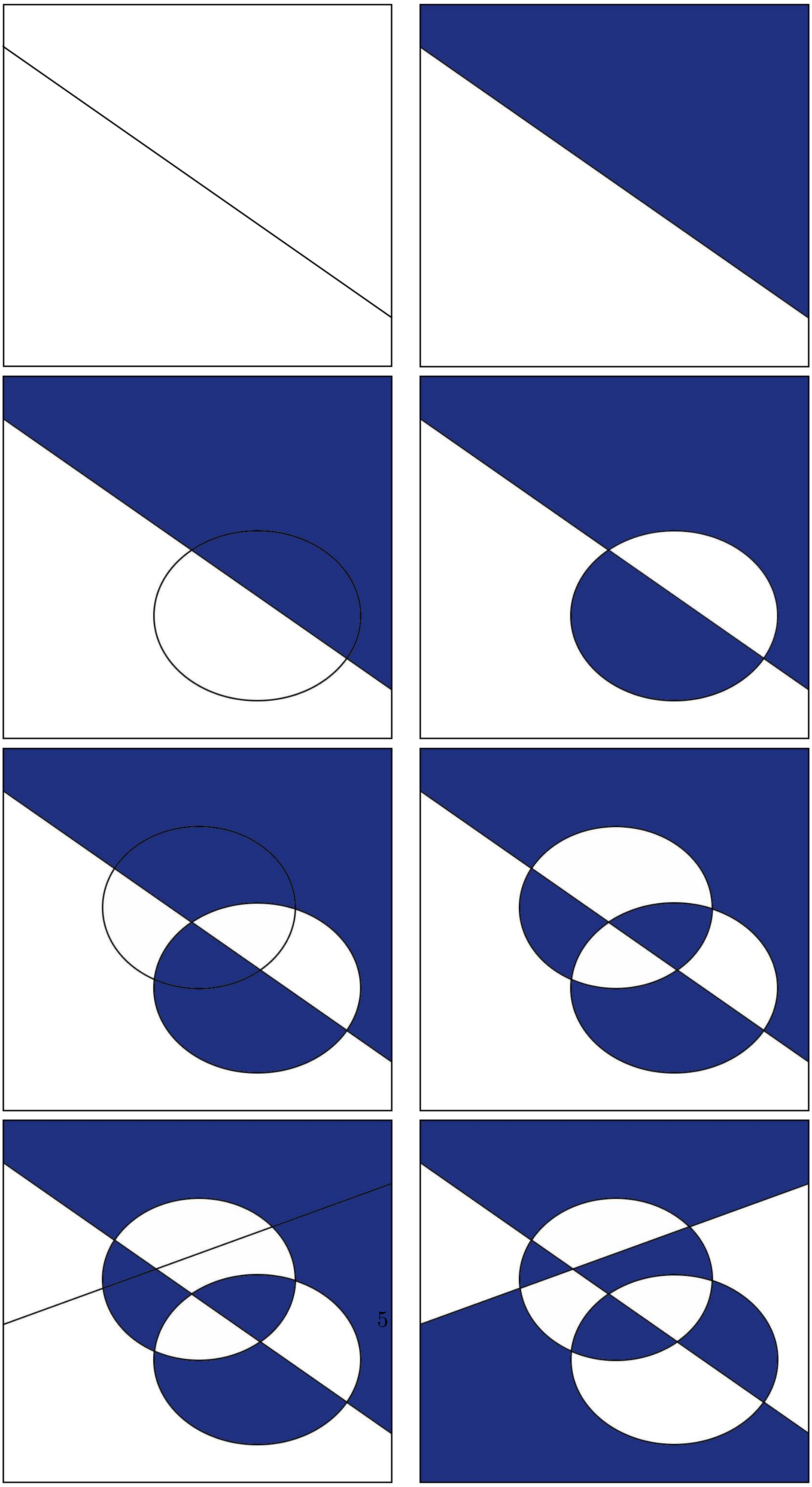

Figure 5: Applying the proof by induction to our example IB-map.